DIRAC OPERATOR WITH CONCENTRATED NONLINEARITY AND

BIFURCATION OF EMBEDDED EIGENVALUES FROM

THE BULK OF THE ESSENTIAL SPECTRUM

A Thesis

by

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ABSTRACT

We study the nonlinear Dirac equation with Soler-type nonlinearity in one dimension (which is called the Gross-Neveu model), with the nonlinearity localized at one and at two points. We study the spectral stability of the solitary wave solutions in these models. As a consequence, we obtain the result that the eigenvalues of the equation with the Soler-type nonlinearity move along the imaginary axis.

We also construct solitary waves under perturbations of the model and look for relations between components of solitary waves in light of the techniques which we use for analyzing the Gross-Neveu model.

We apply the same analysis to the Dirac equation with the concentrated nonlinearity of the same type as in the massive Thirring model. We find the same spectrum of linearization at solitary waves as that in the nonlinear Dirac equation with Soler-type nonlinearity.

DEDICATION

To my siblings

CONTRIBUTORS AND FUNDING SOURCES

This work was supervised by a thesis committee consisting of Professor Andrew Comech and Professor Peter Howard of the Department of Mathematics and Professor Artem G. Abanov of the Department of Physics & Astronomy.

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1. INTRODUCTION

This thesis is a study on the nonlinear Dirac equation with the perturbed Soler-type with concentrated nonlinearity at one point. Since the same notions as in the Schrödinger wave equation are used to consider the relativistic wave equation for the electron [15], which is called the Dirac equation, it is essential to understand the fundaments of the Schrödinger wave equation.

In order to consider and understand the Schrödinger wave equation from mathematical and physical perspectives, it is necessary to see where the equation comes from and how it is developed. Thus, we initially look at the background of the equation, i.e., we roughly consider the concepts of "Quantum Mechanics".

Quantum Mechanics, which is one of the fundamental theories of physics, considers and portrays nature using the smallest scale of material, in other words, atomic notions. In this theory, mathematical description plays a key role since the theory became apparent after mathematical formulations had been developed [17].

Before the 20th century, the existence of many kinds of atomic concepts was shown. But, at that time, people believed that think that many of them could be understandable using the terms of "Classical Physics". However, it was realized that Classical Physics concepts come short to understand and explain outcomes of experiments after the development of the atomic model and some discoveries such as X-rays and radioactivity [17]. To overcome this problem, Planck explained the black body spectrum using the term "quanta" at the beginning of 20th century [8]. Quanta are small indivisible amounts of energy. After Planck, some scientists such as Einstein and Debye worked with this "quantum" concept, which is now called the old quantum theory [8, 17]. Nevertheless, the theory did not work on aperiodic systems since only a qualitative description was provided and no one could explain the theory quantitatively until Rutherford's study in 1911 [17].

In order to understand how radiation behaves, we roughly explain one of the most important experiments, which is called the double-slit experiment [8, 17]. Here, we consider a diffraction experiment which is performed by using a light source reaching the screen with two slits through. A diffraction happens in these slits and the resulting wave reaches the photosensitive screen as Figure 1 shows; it can be seen that a large amount of ejected photoelectrons are in the region where the diffraction reaches the maximum. From this, it can be concluded that the light reaches the screen as though it has the dual wave-particle property. In other words, it acts like a wave when passing through the slits. George Paget Thomson, who is a son of J. J. Thomson, conducted this experiment using metal foils with slits and found the interference effect in 1927 [24].

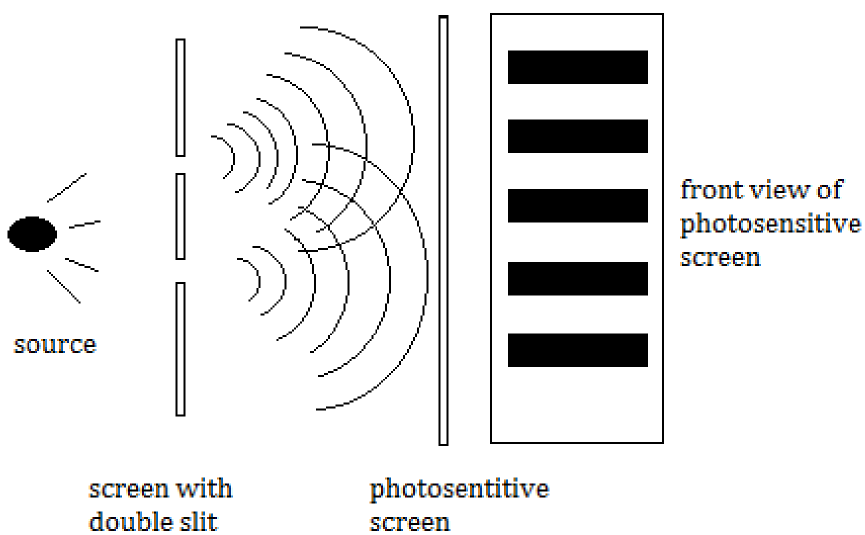

Figure 1: Double slit experiment which shows the wave-like character. Reprinted from [10].

Also, Davisson and Germer made a similar experiment in 1925. When they were considering characteristics of nickel, they incidentally found the wave-like character of electron [11]. They performed this experiment by using the low-energy electron reflecting on a thin film of nickel. Because of the high temperature, the film heated up and separated into large crystal parts after cooling. In the light of de Broglie's thesis, in which was speculated that electron could be represented by a wave, they discovered wave-like character of electron in the experiment [11].

Furthermore, when matter is used in the experiment with double-slit, one obtains similar results, showing that radiation has a particle-like character. Also, according to Einstein's statement, quanta act without disintegrating and they are formed and amortized as complete elements [25].

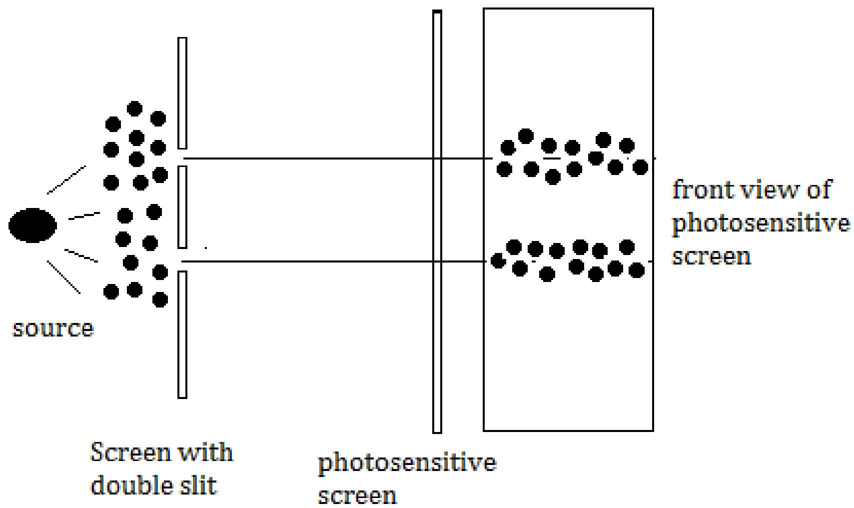

Figure 2: Double slit experiment which shows the particle-like character. Reprinted from [10].

This paper is organized as follows. In this introduction chapter, we consider how the Schrödinger wave equation was developed by looking at some conditions and properties. Subsequently, we begin to study the nonlinear instead of the linear theory because the theory of electromagnetism seems to have a nonlinear nature. For example, the Bohr transitions do not have counterparts in a linear theory, while they are quite natural in a nonlinear one, when superpositions of different eigenfunctions no longer form a solution. While a system such as Schrödinger-Coulomb (or more precisely Dirac-Maxwell) is too complicated, one starts with considering simpler models with nonlinear self-interaction, such as nonlinear Schrödinger or nonlinear Klein-Gordon equations [7]. Moreover, as an introduction, we examine the linear instability of stationary solutions to a nonlinear wave equation by using Derrick's Theorem and Vakhitov-Kolokolov criterion for the nonlinear Schrödinger wave equation [7].

In Chapter II, we consider the studies on the Dirac equation with the Soler-type concentrated nonlinearity at one point focusing on the instability of the spectrum of the linearized operator under perturbations in odd-even-odd-even subspaces. In Chapter III, shifting Dirac-delta functions δ on the nonlinearity of the Soler-model, which we consider in Chapter II, we analyze solitary waves under perturbations. Finally, in Chapter IV, changing Pauli matrix β on the nonlinearity of the nonlinear Dirac equation in Chapter II, in other words, on the massive Thirring model with concentrated at one point, we look for the spectrum of the linearization operator.

1.1. Development of the Schrödinger Wave Equation

In this section, we analyze the formation and the development stages of the Schrödinger wave equation which represents the motion of a particle, quantitatively. In order to consider this equation correctly, several assumptions such as boundary and continuity conditions have to be taken into consideration.

Also, we must pay attention to whether the equation which we write down will provide the following properties of the wave function ψ [17]:

1. In order to explain the consequences of diffraction experiments, it can interfere with itself.
2. It is large in the possible region where the particle exists, it is small elsewhere.
3. It does not represent the statistical distribution of some quanta; rather it describes the behavior of a single particle.

Here, we focus on describing these properties mathematically and need to know how to find the momentum p and the amount of energy E in quanta. First, the energy is calculated by multiplying the frequency ν of the radiation by Planck's constant h:

$$E = h\nu$$

Also, the momentum p of the particle and the length λ of the wave are related as the following equality:

$$p = \frac{h}{\lambda}$$

Substituting the equality $\hbar = \frac{h}{2\pi}$, we get the following equations:

$$p = \hbar k \qquad k = \frac{2\pi}{\lambda} \qquad E = \hbar\omega \qquad \omega = 2\pi\nu. \tag{1.1}$$

Looking at some diffraction experiments such as the experiment of Davisson and Germer, $\psi(x,t)$ will be in the form of one of the following trigonometric functions [17]:

$$\cos(kx - \omega t) \qquad \sin(kx - \omega t) \qquad e^{\mathrm{i}(kx-\omega t)} \qquad e^{-\mathrm{i}(kx-\omega t)} \tag{1.2}$$

or linear combination of them since a particle travels in the positive x direction. Using the relation between the momentum p and the energy E, $E = \frac{p^2}{2m}$, we have $\omega = \frac{\hbar k^2}{2m}$ from the equations (1.1).

This idea belonged to Schrödinger and on the suggestion of Debye, Schrödinger gave a seminar on the Broglie's PhD Thesis in 1925 since it had many relations to his studies in Zeit. f. Physik 12, 13, 1922 [14]. He desired to discover the structure of traveling waves during the refraction to move in the Bohr orbits. However, Debye thought that Schrödinger's idea on the wave-particle duality was childish, and he stated that Schrödinger needed to find a wave equation to approach waves properly. In the next several weeks, Schrödinger gave a seminar and opened his talk by announcing that he had found a wave equation [14].

Now, if we consider the differential equation

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \gamma\frac{\partial^2\psi}{\partial x^2}, \tag{1.3}$$

then it can be seen that the last two functions of (1.2) are solutions of the equation for $\gamma = \pm\frac{\hbar^2}{2m}$.

Therefore, the differential equation (1.3) becomes the following equation:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{1.4}$$

Replacing the equalities $\frac{\partial\psi}{\partial t} = -\mathrm{i}\omega\psi$ and $\frac{\partial^2\psi}{\partial x^2} = -k^2\psi$, it can be concluded that the left side of equation (1.4) represents $E\psi$ and the right side represents $\frac{p^2}{2m}\psi$.

Until now, we considered the one-dimensional form of the Schrödinger wave equation. Now, we extend this form to 3 dimensions. We firstly express the previous terms such as the momentum p in a vectoral form:

$$\mathbf{p} = \hbar\mathbf{k}, \qquad k = |\mathbf{k}| = \frac{2\pi}{\lambda}, \tag{1.5}$$

where **k** represents the propagation vector. Using the same process, we write the exponential function $e^{(\mathbf{k}.\mathbf{r}-\omega t)}$ where the vector **r** represents the position of the particle. Therefore, in 3 dimensions, the Schrödinger wave equation is written as:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi \tag{1.6}$$

As in the one-dimensional case, equations (1.5) and (1.6) provide the relation

$$E = \frac{\mathbf{p}^2}{2m}, \tag{1.7}$$

and then using this relation, it can be said that the operators $\mathrm{i}\hbar\frac{\partial}{\partial t}$ and $-\mathrm{i}\hbar\nabla$ symbolize the energy E and the momentum p, respectively:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = E\psi \tag{1.8}$$

Definition 1. *An equation in the form* (1.8) *is an* ***eigenvalue equation*** *where* ψ *is an* ***eigenfunction*** *with the corresponding* ***eigenvalue*** E.

Moreover, the influence of external forces such as electrostatic and gravitational forces must be observed. We must be aware of the fact that these kinds of forces can be assembled in a force **F** [17]: $\mathbf{F}(\mathbf{r},t) = -\nabla V(\mathbf{r},t)$. Including the potential energy V to the equality (1.6), we get the total energy E

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r},t) \tag{1.9}$$

and then, the generalized Schrödinger wave is

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r},t). \tag{1.10}$$

Furthermore, since quantum mechanics does not give details regarding the circumstances during the specific movement, using the probability theory, a prediction can be made about the experimental results [17, 19]. The wave function $\psi(\mathbf{r},t)$ can therefore be interpreted as the probability amplitude; however, $\psi(\mathbf{r},t)$ can be complex. Since ψ represents the probability, we need a positive and real term. Then, taking the square of the norm of the function ψ, we define the probability density as the product $\psi^*\psi$ where ψ^* is the complex conjugate of ψ [14].

$$P(\mathbf{r},t) = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$$

As $|\psi(\mathbf{r},t)|^2$ represents the probability, the value of $|\psi(\mathbf{r},t)|^2$ in whole region must be 1:

$$\int |\psi(\mathbf{r},t)|^2 \, d^3\mathbf{r} = 1$$

Hence, it can be concluded that the wave function ψ is normalized.

Now, we return to develop the properties of the Schrödinger wave equation. Assume that the potential energy V does not depend on time. Replacing the equality $\psi(\mathbf{r},t) = u(\mathbf{r})v(t)$ to equation

(1.10), we obtain

$$\mathrm{i}\hbar u(\mathbf{r})\frac{dv(t)}{dt} = -\frac{\hbar^2}{2m}v(t)\nabla^2 u(\mathbf{r}) + V(\mathbf{r})u(\mathbf{r})v(t)$$

and then

$$\frac{\mathrm{i}\hbar}{v(t)}\frac{dv(t)}{dt} = -\frac{\hbar^2}{2m}\frac{1}{u(\mathbf{r})}\nabla^2 u(\mathbf{r}) + V(\mathbf{r}) \tag{1.11}$$

dividing both sides by $u(\mathbf{r})v(t)$. Since the left side of the equality (1.11) depends only on time and the right side depends only on the position, it can be said that the equality is constant. By the equality $E\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}$, we have:

$$-\frac{\hbar^2}{2m}\nabla^2 u(\mathbf{r}) + V(\mathbf{r})u(\mathbf{r}) = Eu(\mathbf{r})$$

$$\mathrm{i}\hbar\frac{dv(t)}{dt} = Ev(t)$$

It follows that E is constant where $\frac{\mathrm{i}\hbar}{v(t)}\frac{dv(t)}{dt} = E$ and then the solution of the wave equation with the potential energy $V(r)$ is in the following form:

$$\psi(\mathbf{r}, t) = u(\mathbf{r})e^{-\mathrm{i}\frac{Et}{\hbar}}, \tag{1.12}$$

and then it is stationary since $|\psi|^2$ is constant in time.

Also, we need to examine the Schrödinger wave equation by considering the boundary conditions of the potential energy and the position. In other words, looking at the Figure 3, we analyze the solution $\psi(x)$ where $0 < x < x_0$ and $0 < V < \infty$ (in one dimension) [17]:

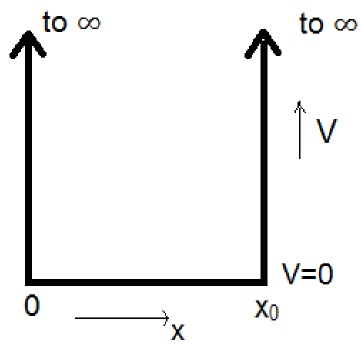

Figure 3: Boundary conditions of the potential energy and the position. Reprinted from [17].

1. For $x \leq 0$, $x \geq x_0$ and $V(x) = \infty$, it is impossible to find the electron near the point x, i.e., the probability $\psi = 0$;

2. for $x = 0$, $x = x_0$, i.e., at the boundaries, we have $\psi(0) = \psi(x_0) = 0$ using the continuity conditions;

3. for $0 < x < x_0$, assuming that V is constant, the value of V can be taken to zero. Using this, the equation will become
$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + E\psi(x) = 0. \tag{1.13}$$

Let us denote $k^2 = \frac{2mE}{\hbar^2}$ where E is positive. In the case that E is negative, there is no solution using the boundary conditions above. We find the solution in the following trigonometric form:

$$\psi(x) = A\sin kx + B\cos kx$$

Looking at the boundaries, we get that $\psi(0) = B = 0$ and $\psi(x_0) = A\sin kx_0 + B\cos kx_0 = 0$. It follows that $\sin kx_0 = 0$ for nonzero A, i.e., $kx_0 = \pi n$ for $n \in \mathbb{Z}^+$. We can represent k as the sequence $k_n = \frac{\pi n}{x_0}$ since the value of k depends on n. Therefore, the solution becomes

$$\psi_n(x) = A_n \sin k_n x \qquad \text{where} \qquad k_n = \frac{\sqrt{2mE}}{\hbar}.$$

Hence, we can rewrite the energy E as the sequence

$$E_n = \frac{\hbar^2 {k_n}^2}{2m}. \tag{1.14}$$

Using the fact that ψ is normalized in entire region:

$$\int_0^{x_0} |\psi(x)|^2 \, dx = \int_0^{x_0} {A_n}^2 \sin^2 k_n x \, dx = 1,$$

it can be found that $A_n = \sqrt{\frac{2}{x_0}}$ and then $\psi_n(x) = \sqrt{\frac{2}{x_0}} \sin k_n x$.

From this computation, it can be concluded that the solution of the Schrödinger wave equation exists only in the case that the energy takes the discrete value [19].

In (1.14), the terms n are named *quantum numbers* and the discrete energy E values are named *energy levels*.

Also, we said an electron in *the quantum state n* if the wave function of the electron has n value [14].

Unlike classical mechanics, in quantum mechanics, the kinetic energy is different from zero in the ground state which represents the quantum energy in the case $n = 1$ [19].

1.2. Instability of Stationary Localized Solutions

In this section, we look at the nonlinear Klein-Gordon equation:

$$-\ddot{\psi} = -\Delta\psi + g(\psi), \qquad \psi(x,t) \in \mathbb{R}, \qquad x \in \mathbb{R}^n, \qquad n \geq 1, \tag{1.15}$$

with the smooth nonlinearity $g(\psi)$ which satisfies $g(0) = 0$. We study the instability of stationary solutions to equation (1.15). We write the equation in the form of the Hamiltonian system:

$$\dot{\varphi} = -\delta_\psi E, \qquad \dot{\psi} = \delta_\varphi E, \tag{1.16}$$

with the Hamiltonian of the form

$$E(\psi,\varphi) = \int_{\mathbb{R}^n} \Big(\frac{\varphi^2}{2} + \frac{|\nabla\psi|^2}{2} + G(\psi) \Big)\, dx,$$

where $G(t) = \int_0^t g(s)ds$ [7]. We focus on the fact that there are no stable localized stationary solutions in dimension $n \geq 3$ [9]. This fact comes from Derrick's theorem; below, we denote the localized stationary solution $\psi(x,t)$ in the form $\theta(x)$.

Theorem 1.1 ([9])**.** *Let $\theta(x)$ be the stationary solution of equation* (1.15) *in dimension n. Then, $\theta(x)$ does not minimize $\int_{\mathbb{R}^n} \frac{|\nabla\psi|^2}{2} + G(\psi)\, dx$.*

Proof. We closely follow [9]. Let $T(\theta) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\theta|^2 dx$ and $V(\theta) = \int_{\mathbb{R}^n} G(\theta)\, dx$. The Hamiltonian system (1.16) takes the form

$$0 = \dot{\varphi} = -\frac{\delta E}{\delta \psi}(\theta, 0) \qquad 0 = \dot{\psi} = \frac{\delta E}{\delta \varphi}(\theta, 0).$$

We consider the family $\theta_\lambda(x) = \theta\big(\frac{x}{\lambda}\big)$. Therefore,

$$T(\theta_\lambda) = \int_{\mathbb{R}^n} \frac{|\nabla\theta_\lambda(x)|^2}{2}\, dx = \lambda^n \int \frac{|\nabla\theta\big(\frac{x}{\lambda}\big)|^2}{2}\, d^n\Big(\frac{x}{\lambda}\Big) = \lambda^n \int \frac{1}{\lambda^2}\frac{|\nabla\theta_\lambda(y)|^2}{2}\, dy = \lambda^{n-2} T(\theta),$$

$$V(\theta_\lambda) = \int_{\mathbb{R}^n} G(\theta_\lambda(x))\, dx = \lambda^n \int G(\theta\big(\frac{x}{\lambda}\big))\, d^n\Big(\frac{x}{\lambda}\Big) = \lambda^n V(\theta),$$

and then,

$$0 = \Big\langle \frac{\delta E}{\delta\psi}(\theta,0), \frac{\partial\theta_\lambda}{\partial\lambda}\Big|_{\lambda=1} \Big\rangle = \frac{\partial E}{\partial\lambda}(\theta_\lambda, 0)\Big|_{\lambda=1} = (n-2)T(\theta) + nV(\theta). \tag{1.17}$$

The equality (1.17) is called Pohozaev's identity [16]. This leads to the equality:

$$\frac{\partial^2 E}{\partial\lambda^2}(\theta_\lambda)\Big|_{\lambda=1} = (n-2)(n-3)T(\theta) + n(n-1)V(\theta) = -2(n-2)T(\theta).$$

It can be seen that $\frac{\partial^2 E}{\partial \lambda^2}(\theta_\lambda)\Big|_{\lambda=1}$ is negative for $n \geq 3$. It follows that $T(\theta_\lambda) + V(\theta_\lambda)$ does not have a minimum at $\lambda = 1$. $\square$

In this theorem, we conclude that the stationary solution $\theta(x)$ is unstable for $n \geq 3$. However, one cannot make any conclusion about instability of the solutions in the dimensions $n = 1$ and $n = 2$ [7].

In order to consider the instability of stationary solutions in any dimensions, we make some changes on Derrick's theorem:

Lemma 1 ([12, 5])**.** *If $\theta \in H^\infty(\mathbb{R}^n)$ is a stationary solution of equation* (1.15) *for any $n \geq 1$. Then θ is linearly unstable.*

Proof. Let us look at the solution in the form $\psi(x,t) = \theta(x) + \rho(x,t)$ where $\rho(x,t)$ is a small perturbation. Since θ does not depend on time, it satisfies

$$-\Delta\theta + g(\theta) = 0. \tag{1.18}$$

Since $\lim_{|x|\to\infty} \theta(x) = 0$, it can be said that $\frac{\partial \theta}{\partial x_1} = 0$ somewhere. Equation (1.15) takes the form

$$-\ddot{\rho}(x,t) = -\Delta\theta(x) - \Delta\rho(x,t) + g(\theta(x) + \rho(x,t)),$$

$$\ddot{\rho}(x,t) = -\Delta\rho(x,t) + g'(\theta(x))\rho(x,t),$$

and then we get the following equality:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Delta - g'(\theta) & 0 \end{bmatrix}\begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -L & 0 \end{bmatrix}\begin{bmatrix} \rho \\ \dot{\rho} \end{bmatrix}. \tag{1.19}$$

Let us denote $A = \begin{bmatrix} 0 & 1 \\ \Delta - g'(\theta) & 0 \end{bmatrix}$. We will show that the matrix A has a positive eigenvalue.

Taking the first derivative of the equality (1.18), we have $-\Delta \frac{\partial \theta}{\partial x_1} + g'(\theta)\frac{\partial \theta}{\partial x_1} = 0$ and it follows that $\frac{\partial \theta}{\partial x_1}$ is the eigenvector of the operator $L = -\Delta + g'(\theta)$ corresponding the eigenvalue $\lambda = 0$ of the operator A. However, since $\frac{\partial \theta}{\partial x_1}$ vanishes somewhere, there is another eigenvector $\chi \in H^\infty(\mathbb{R}^n)$ corresponding to negative eigenvalue of L:

$$L\chi = -a^2\chi \qquad \text{with some} \quad a > 0.$$

Hence, we obtain that

$$A \begin{bmatrix} \chi \\ \pm a\chi \end{bmatrix} = \pm a \begin{bmatrix} \chi \\ \pm a\chi \end{bmatrix},$$

and then the matrix has eigenvectors $\begin{bmatrix} \chi \\ \pm a\chi \end{bmatrix}$ corresponding to the eigenvalues $\pm a \in \mathbb{R}$. Therefore, we conclude that the stationary solution θ is linearly unstable. $\square$

1.3. Solitary waves in one dimension

Let us consider the nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + g(|\psi|^2)\psi, \qquad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}, \tag{1.20}$$

where g is a smooth, real-valued function and $\psi(x,t) \in \mathbb{C}$. Let us construct solitary wave solutions.

Definition 2. *A solitary wave is a solutions of equation* (1.20) *of the form*

$$\psi(x,t) = \phi_\omega(x)e^{-\mathrm{i}\omega t}$$

where $\phi_\omega \in H^1(\mathbb{R})$ *and* $\omega \in \mathbb{R}$.

Substituting a solitary wave into equation (1.20), we get the stationary equation

$$\omega\phi(x) = \phi''(x) + g(\phi^2)\phi, \tag{1.21}$$

and we rewrite (1.21) as below:

$$\phi''(x) = g(\phi^2)\phi - \omega\phi(x) = -\frac{\partial}{\partial\phi}\frac{\omega\phi^2 - G(\phi^2)}{2} \tag{1.22}$$

where $G(s) = \int_0^s g(v)dv$. We examine the relation (1.22) by defining the potential energy as

$$V_\omega(\phi) := \frac{\omega\phi^2 - G(\phi^2)}{2}$$

such that x represents the time and ϕ represents the position of a point particle. Therefore, the relation (1.22) provides the mechanical energy, which is given by $\epsilon(\phi) = |\phi|^2 + V_\omega(\phi)$, and from this, we can see that ϵ does not depend on time (represented by x) for a solution $\phi(x)$ [7].

1.4. Vakhitov-Kolokolov Criterion

Previously, we considered the instability of stationary localized solutions of the nonlinear wave equation by means of Derrick's theorem and linear stability analysis. We now demonstrate that there can exist stable solitary wave solutions.

We constructed solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ in one dimension by considering the nonlinear Schrödinger wave equation (1.20). Now, we suppose that $g(0) = 0$ and choose that the solution ϕ which satisfies the stationary equation (1.21) is positive. Therefore, ω becomes negative.

Adding small perturbation $\rho(x,t) \in \mathbb{C}$ to the solitary wave equation, we look at the solution in the form $\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-\mathrm{i}\omega t}$. We denote $\rho(x,t) = R(x,t) + \mathrm{i}S(x,t)$, with R, S real-valued. Therefore, substituting the Ansatz into the Schrödinger wave equation (1.20), we get the following relation (see e.g. [5, 7]):

$$w(\phi + R + \mathrm{i}S) + \mathrm{i}\dot{R} - \dot{S} = -\phi'' - R'' - \mathrm{i}S'' + g(|\phi + R + \mathrm{i}S|^2)(\phi + R + \mathrm{i}S).$$

Separating the imaginary and real parts and using the stationary equation (1.21), the following system can be obtained:

$$wS + \dot{R} = -S'' + g(|\phi|^2)S$$

$$wR - \dot{S} = -R'' + g(|\phi|^2)R + g'(|\phi|^2)2\phi R,$$

and putting this system into the following matrix system:

$$\frac{\partial}{\partial t}\begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 0 & L_-(\omega) \\ -L_+(\omega) & 0 \end{bmatrix}\begin{bmatrix} R \\ S \end{bmatrix}$$

where

$$L_-(\omega) = -\frac{\partial^2}{\partial x^2} + g(\phi^2) - \omega, \qquad L_+(\omega) = -\frac{\partial^2}{\partial x^2} + g(\phi^2) + 2\phi^2 g'(\phi^2) - \omega.$$

Hence, we can write the linearization at a solitary wave as below:

$$\frac{\partial \boldsymbol{\rho}}{\partial t} = \boldsymbol{J}\boldsymbol{L}(\omega)\boldsymbol{\rho}, \qquad \boldsymbol{\rho}(x,t) = \begin{bmatrix} \operatorname{Re}\rho(x,t) \\ \operatorname{Im}\rho(x,t) \end{bmatrix}, \tag{1.23}$$

where $\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\boldsymbol{L}(\omega) = \begin{bmatrix} L_+(\omega) & 0 \\ 0 & L_-(\omega) \end{bmatrix}$.

Moreover, following [20], we show that the spectrum of $\boldsymbol{J}\boldsymbol{L}$ stays on the axes only: $\sigma(\boldsymbol{J}\boldsymbol{L}) \subset$

$\mathbb{R} \cup \mathrm{i}\mathbb{R}$. Let λ be an eigenvalue of $\boldsymbol{JL}$. We prove it by using the spectrum of $(\boldsymbol{JL})^2$:

$$\lambda^2 \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} -L_-(\omega)L_+(\omega) & 0 \\ 0 & -L_+(\omega)L_-(\omega) \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}$$

We notice that both L_+ and L_- are self-adjoint. Since $L_-(\omega)\phi_\omega = 0$ while $\phi_\omega > 0$ is not equal to zero, it corresponds to the smallest eigenvalue, and then L_- is positive-definite. Hence, using the fact that $L_-^{1/2}L_+^{1/2}L_-^{1/2}$ is self-adjoint,

$$\sigma_d((\boldsymbol{JL})^2) - \{0\} = \sigma_d(L_-L_+) - \{0\} = \sigma_d(L_+L_-) - \{0\} = \sigma_d(L_-^{1/2}L_+^{1/2}L_-^{1/2}) - \{0\} \subset \mathbb{R}.$$

Therefore, we obtain that $\lambda^2 \in \mathbb{R}$ for any $\lambda \in \sigma_d(\boldsymbol{JL})$.

Furthermore, we now find the condition on ω so that nonzero real part eigenvalues of the linearized equation exist. We know that this kind of eigenvalues can be found only on the real axis. Moreover, it can be shown that $\lambda = 0$ is in the set of discrete spectrum of $\boldsymbol{JL}$ since $L_-\phi_\omega = 0$: for all ω corresponding to solitary waves,

$$\boldsymbol{JL} \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix} = 0, \qquad \boldsymbol{JL} \begin{bmatrix} -\frac{\partial \phi_\omega}{\partial \omega} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix}.$$

Considering even functions in x, we can say that the operator $\boldsymbol{JL}$ has the at least 2-dimensional null space. If there is ζ such that

$$\boldsymbol{JL}\zeta = \begin{bmatrix} \frac{\partial \phi_\omega}{\partial \omega} \\ 0 \end{bmatrix},$$

then $\begin{bmatrix} \frac{\partial \phi_\omega}{\partial \omega} \\ 0 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} \phi_\omega \\ 0 \end{bmatrix}$ which is the vector in the null space of the adjoint to $\boldsymbol{JL}$.

Therefore,

$$\big\langle \phi_\omega, \frac{\partial \phi_\omega}{\partial \omega} \big\rangle = \frac{\partial ||\phi_\omega||^2_{L_2}}{\partial \omega^2} = 0.$$

As a result of a serious study [6], it can be said that when $\frac{\partial ||\phi_\omega||^2_{L_2}}{\partial \omega^2} > 0$, there exists two eigenvalues $\pm\lambda \in \mathbb{R}$ and then it follows a linear instability of solitary wave.

However, if $\frac{\partial ||\phi_\omega||^2_{L_2}}{\partial \omega^2}$ is negative, Vakhitov-Kolokolov stability criterion, which is considered in [5], claims that no nonzero real eigenvalues can be found for the nonlinear Schrödinger equation.

2. ON THE NONLINEAR DIRAC EQUATION WITH CONCENTRATED NONLINEARITY

In 1928, Paul Dirac worked on an alternative relativistic wave equation since there were some problems such as negative energy solutions, negative particle density with the Klein-Gordon equations which is

$$-\frac{\partial^2\Psi}{\partial t^2} = -\nabla^2\Psi + m^2\Psi,$$

and he found it with all particle densities are positive [22]. Using the classical wave equation and the equalities (1.6) and (1.8), he created a different equation as below:

$$E\Psi = (\vec{\alpha}.\vec{p} + \beta m)\Psi = \mathrm{i}\frac{\partial\Psi}{\partial t} \tag{2.1}$$

We can write equation (2.1) more clearly:

$$\left(-\mathrm{i}\alpha_x\frac{\partial\Psi}{\partial x} - \mathrm{i}\alpha_y\frac{\partial\Psi}{\partial y} - \mathrm{i}\alpha_z\frac{\partial\Psi}{\partial z} + \beta m\right)\Psi = \mathrm{i}\frac{\partial\Psi}{\partial t} \tag{2.2}$$

Also, a free particle provides the equality $E^2 = \vec{p}^2 + m^2$, in other words the Klein-Gordon equation, by squaring (2.2), we get the following equality [13]:

$$\left(-{\alpha_x}^2\frac{\partial^2\Psi}{\partial x^2} - {\alpha_y}^2\frac{\partial^2\Psi}{\partial y^2} - {\alpha_z}^2\frac{\partial^2\Psi}{\partial z^2} + \beta^2 m^2\right)\Psi = -\frac{\partial^2\Psi}{\partial t^2}$$

with ${\alpha_x}^2 = {\alpha_y}^2 = {\alpha_z}^2 = \beta^2 = 1$, $\alpha_i\beta + \beta\alpha_i = 0$, and $\alpha_i\alpha_k + \alpha_k\alpha_i = 0$ for $i \neq k$. An appropriate option is placed on the Pauli matrices which are

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}$$

with $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this chapter, we consider the nonlinear Dirac equation with concentrated nonlinearity and the behavior of the stability properties of solitary waves under the perturbation of the nonlinearity. Also, the Dirac equation in the nonlinear form is called as the Soler model.

2.1. Solitary Waves

Here, we introduce the nonlinear Dirac equation with Soler-type nonlinearity concentrated at one point, which has been introduced in [4]:

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\mathrm{i}\alpha \frac{\partial \psi}{\partial x} + \beta m \psi - \delta(x)|\psi^* \beta \psi|^k \beta \psi, \qquad m > 0. \tag{2.3}$$

The solitary waves in this model and their linear stability are considered in the preprint [3]. We give the necessary details. For the solitary wave solutions $\psi(x,t) = \phi(x)e^{-\mathrm{i}\omega t}$ where $(x,t) \in \mathbb{R} \times \mathbb{R}$ and $\omega \in (0, m)$, we get

$$\omega \phi = -\mathrm{i}\alpha \frac{\partial \phi}{\partial x} + \beta m \phi - \delta(x)|\phi^* \beta \phi|^k \beta \phi. \tag{2.4}$$

Choosing $\alpha = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we study on the solitary waves for

$$\phi_\omega = \begin{bmatrix} a \\ b \operatorname{sgn} x \end{bmatrix} e^{-\kappa|x|} \tag{2.5}$$

where $\kappa = \sqrt{m^2 - \omega^2}$. Substituting ϕ_ω into equation (2.4), we obtain the relation

$$(m-\omega) a e^{-\kappa|x|} + \frac{\partial}{\partial x} b \operatorname{sgn} x e^{-\kappa|x|} = \delta(x) |a|^{2k} a e^{-\kappa|x|}. \tag{2.6}$$

We note that the product $\delta(x) \operatorname{sgn} x$ is identically zero. At this point, we have the following equality:

$$\begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} \begin{bmatrix} a \\ b \operatorname{sgn} x \end{bmatrix} e^{-\kappa|x|} = |a|^{2k} \delta(x) \begin{bmatrix} a \\ 0 \end{bmatrix} e^{-\kappa|x|}. \tag{2.7}$$

In the case $x > 0$, we have:

$$\begin{bmatrix} m-\omega & -\kappa \\ \kappa & -m-\omega \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0,$$

and then we get the relation between a and b which is given by

$$b = \frac{\kappa}{m+\omega} = \mu a, \qquad \text{where} \qquad \mu := \sqrt{\frac{m-\omega}{m+\omega}}.$$

Using the relation $\frac{\partial}{\partial x} \operatorname{sgn} x = 2\delta(x)$ for the Dirac δ-function and collecting the coefficients at $\delta(x)$ in (2.7), gives the following relation, which is called the jump condition:

$$2b = |a|^{2k} a, \qquad \text{and then} \qquad \mu = \frac{|a|^{2k}}{2}. \tag{2.8}$$

2.2. Spectrum of the linearization operator

In this section, we look for the linearization at a solitary wave using an small perturbation ρ which depends on time, and find the spectrum of the linearization operator in odd-even-odd-even subspaces. We follow [3].

We begin with the analysis of the nonlinear Dirac equation (2.3) using the solution in the form of the Ansatz

$$\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-i\omega t} \in \mathbb{C}^2$$

where $\rho(x,t) = R(x,t) + \mathrm{i}S(x,t) \in \mathbb{C}^2$. If we substitute the solution $\psi(x,t)$ in the form of this Ansatz into equation (2.3), we get the following relations:

$$\begin{aligned} \omega R + \mathrm{i}\omega S + \mathrm{i}\dot{R} - \dot{S} = -\mathrm{i}\alpha\frac{\partial\phi_\omega}{\partial x} - \mathrm{i}\alpha\frac{\partial R}{\partial x} - \mathrm{i}\alpha\frac{\partial \mathrm{i}S}{\partial x} + \beta m\phi_\omega + \beta mR + \beta m\mathrm{i}S \\ -\delta(x)f(\phi_\omega + R + \mathrm{i}S)\beta(\phi_\omega + R + \mathrm{i}S), \end{aligned} \tag{2.9}$$

where $f(\phi) = |\phi^*\beta\phi|$. We note that one can use the linear approximation as below:

$$|(\phi_\omega + R + \mathrm{i}S)^*\beta(\phi_\omega + R + \mathrm{i}S)| \approx |\phi_\omega{}^*\beta(\phi_\omega + 2R)|,$$

and then we arrive at the following system by separating imaginary and real parts of equations (2.9):

$$\dot{R} = -\mathrm{i}\alpha\frac{\partial S}{\partial x} + \beta mS - \delta(x)f(\phi_\omega)\beta S - \omega S, \tag{2.10}$$

$$-\dot{S} = -\mathrm{i}\alpha\frac{\partial R}{\partial x} + \beta mR - \delta(x)f(\phi_\omega)\beta R - 2\delta(x)|\phi_\omega^*\beta R|f'\beta\phi_\omega - \omega R. \tag{2.11}$$

Collecting equations (2.10), (2.11), we write the linearized equation on ρ:

$$\frac{\partial \boldsymbol{\rho}}{\partial t} = \boldsymbol{J}\boldsymbol{L}\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix},$$

where

$$\boldsymbol{L} = \boldsymbol{J}\boldsymbol{\alpha}\frac{\partial \boldsymbol{\rho}}{\partial x} + \boldsymbol{\beta}\boldsymbol{\rho} + (m - \delta(x)f)\boldsymbol{\beta}\boldsymbol{\rho} - 2\delta(x)(\boldsymbol{\phi}^*\boldsymbol{\beta}\boldsymbol{\rho})f'\boldsymbol{\beta}\boldsymbol{\phi} - \omega\boldsymbol{\rho}$$

with

$$\boldsymbol{\alpha} = \begin{bmatrix} \operatorname{Re}\alpha & -\operatorname{Im}\alpha \\ \operatorname{Im}\alpha & \operatorname{Re}\alpha \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \operatorname{Re}\beta & -\operatorname{Im}\beta \\ \operatorname{Im}\beta & \operatorname{Re}\beta \end{bmatrix}, \quad \boldsymbol{\phi} = \begin{bmatrix} \operatorname{Re}\phi \\ \operatorname{Im}\phi \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}.$$

Here, we note that $\boldsymbol{\phi}^*\boldsymbol{\beta}\boldsymbol{\rho} = \phi^*(0)\beta\operatorname{Re}\boldsymbol{\rho}(t,0) = a(0)\operatorname{Re}\rho_1(t,0)$ where ϕ is the solitary wave of the form (2.5).

2.2.1. Vakhitov-Kolokolov Criterion

Although we have the Vakhitov-Kolokolov criterion for the Schrödinger wave equation in order to consider the instability, this idea does not work for the nonlinear Dirac equation because L_- is not positive-definite. However, Vakhitov-Kolokolov condition $\frac{\partial Q}{\partial \omega} = 0$, where the charge $Q(u)$ is given by

$$Q(u) = \int_{\mathbb{R}^n} |u|^2\,dx,$$

has some limited application to Dirac equation.

At this point, we have the equalities

$$\boldsymbol{J}\boldsymbol{L}\boldsymbol{J}\phi_\omega = 0, \qquad \boldsymbol{J}\boldsymbol{L}\partial_\omega\phi_\omega = \boldsymbol{J}\phi_\omega,$$

and considering the Vakhitov-Kolokolov criterion $\frac{\partial Q}{\partial \omega} = 0$, we can say whether there is an L^2-solution ζ to $\boldsymbol{JL}\zeta = \partial_\omega \phi_\omega$. If we compute $Q(\phi_\omega)$, we have:

$$Q(\phi_\omega) = \int (v^2 + u^2)\, dx = (|a|^2 + |b|^2) \int_{\mathbb{R}} e^{-2\kappa|x|}\, dx$$

$$= \frac{|a|^2(1+\mu^2)}{\kappa} = \frac{(2\mu)^{1/k}(1+\mu^2)}{\kappa} = \frac{(2\mu)^{1/k}(1+\mu^2)}{(m+\omega)\mu}.$$

Since $1 + \mu^2 = \frac{2m}{m+\omega}$, we can write

$$Q(\phi_\omega) = \frac{2^{1/k}}{2m} \mu^{(1/k)-1}(1+\mu^2)^2.$$

Taking derivative of $Q(\phi_\omega)$, we have

$$\frac{\partial Q(\phi_\omega)}{\partial \omega} = \frac{2^{1/k}}{2m} \frac{\mu^{(1/k)-2}(1+\mu^2)^2}{k}((1-k)(1+\mu^2) + 4k\mu^2).$$

In the case $\frac{\partial Q}{\partial \omega} = 0$, we obtain the equality

$$1 - k + \mu^2 + 3k\mu^2 = 0,$$

and substituting $\mu = \sqrt{\frac{m-\omega}{m+\omega}}$, it follows that

$$\frac{m-\omega}{m+\omega} = \frac{k-1}{1+3k}, \qquad \frac{\omega}{m} = \frac{1+k}{2k}.$$

Therefore, we determine the critical value of ω as

$$\omega_k = \frac{1+k}{2k} m$$

for $k \geq 1$. As a result, we have the following properties:

- If $0 < k \leq 1$, $\frac{\partial Q(\phi_\omega)}{\partial \omega} < 0$ for $\omega \in (0, m)$;
- if $k > 1$, $\frac{\partial Q(\phi_\omega)}{\partial \omega} > 0$ for $\omega \in (\omega_k, m)$ (according to the Vakhitov-Kolokolov criterion, this gives the linear stability), and $\frac{\partial Q(\phi_\omega)}{\partial \omega} < 0$ for $\omega \in (0, \omega_k)$.

2.2.2. The Spectrum of JL in odd-even-odd-even subspaces

In order to consider the instability of the nonlinear Dirac equation in (2.3), we choose the invariant subspaces for the linearization operator $\boldsymbol{JL}$ and compute its spectrum in each of them. The following lemma from [1] helps us to determine such invariant subspaces for the nonlinear Dirac equation which has the form

$$i\frac{\partial \psi}{\partial t} = -i\alpha\frac{\partial \psi}{\partial x} + \beta g(\psi^* \beta \psi)\psi \tag{2.12}$$

where $\psi(x,t) \in \mathbb{C}^N$, and $x \in \mathbb{R}^n$.

We look for the spectrum of the linearization operator on odd-even-odd-even subspaces. In the previous section, we found that the linearization at a solitary wave is represented by the operator

$$\boldsymbol{JL} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix},$$

where

$$\begin{aligned} L_- &= \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - 2\mu\delta(x)\sigma_3, \\ L_+ &= \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - 2\mu\delta(x)\sigma_3 - 4k\mu\delta(x)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.13}$$

From the relation $\boldsymbol{JL}\Psi = \lambda\Psi$ we find that the eigenvector $\Psi(x)$ corresponding to the eigenvalue $\lambda = \mathrm{i}\Lambda$ is

$$\Psi(x) = A \begin{bmatrix} M \operatorname{sgn} x \\ m - \omega - \Lambda \\ \mathrm{i}M \operatorname{sgn} x \\ \mathrm{i}(m - \omega - \Lambda) \end{bmatrix} e^{-M|x|} + B \begin{bmatrix} N \operatorname{sgn} x \\ m - \omega + \Lambda \\ -\mathrm{i}N \operatorname{sgn} x \\ -\mathrm{i}(m - \omega + \Lambda) \end{bmatrix} e^{-N|x|}$$

and $M = \sqrt{m^2 - (\omega + \Lambda)^2}$, $N = \sqrt{m^2 - (\omega - \Lambda)^2}$.

If we look at the jump condition, i.e., the coefficients at the δ-function in the following matrix-vector product

$$\begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix} \Psi = 0$$

where $L = \begin{bmatrix} m - \omega & \partial_x \\ -\partial_x & -m - \omega \end{bmatrix}$, then we get the system

$$\begin{aligned} -MA + NB + \mu(A(m - \omega - \Lambda) - B(m - \omega + \Lambda)) = 0 \\ MA + NB - \mu(A(m - \omega - \Lambda) + B(m - \omega + \Lambda)) = 0 \end{aligned}$$

and it follows that

$$\begin{bmatrix} -M + \mu(m - \omega - \Lambda) & N - (m - \omega + \Lambda) \\ M - \mu(m - \omega - \Lambda) & N - (m - \omega + \Lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, since $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have:

$$\det \begin{bmatrix} -M + \mu(m - \omega - \Lambda) & N - (m - \omega + \Lambda) \\ M - \mu(m - \omega - \Lambda) & N - (m - \omega + \Lambda) \end{bmatrix} = -2(M - \mu(m - \omega - \Lambda))(N - (m - \omega + \Lambda)) = 0.$$

If we have the relation $M - \mu(m - \omega - \Lambda) = 0$, we get:

$$\sqrt{m^2 - (\omega + \Lambda)^2} - \sqrt{\frac{m - \omega}{m + \omega}}(m - \omega - \Lambda) = 0,$$

$$m^2 - (\omega + \Lambda)^2 = \frac{m - \omega}{m + \omega}(m - \omega - \Lambda)^2,$$

$$m + \omega + \Lambda = \frac{m - \omega}{m + \omega}(m - \omega - \Lambda),$$

$$(m + \omega + \Lambda)(m + \omega) = (m - \omega)(m - \omega - \Lambda),$$

$$(m + \omega)^2 - (m - \omega)^2 = -2m\Lambda,$$

$$\Lambda = -2\omega.$$

If instead the relation $N - (m - \omega + \Lambda) = 0$ holds, we obtain $\Lambda = 2\omega$. Therefore, we have the eigenvectors $\Psi(x)$ corresponding to the eigenvalues $\lambda = \pm 2\omega \mathrm{i}$ which are present in the spectrum of $\boldsymbol{JL}$.

2.3. Bifurcation from the essential spectrum under perturbations

In this section, we write out the eigenfunctions in odd-even-odd-even subspaces by classifying the eigenvalues and look for the behavior of the eigenvalues under the perturbations.

When we consider the operators L_- and L_+ in (2.13), we see that both are invariant in the space of odd-even functions. Furthermore, we note that

$$\sigma\left(\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}\Bigg|_{odd-even-odd-even}\right) = \Big\{ \pm \mathrm{i}a; a \in \sigma\big(L_-|_{odd-even}\big)\Big\}$$

since $L_+|_{odd-even} = L_-$. In the case $\Lambda_0 = 2\omega$, we have $\phi(x) = \begin{bmatrix} \eta \operatorname{sgn} x \\ m + \omega \end{bmatrix} e^{-\kappa|x|}$ with $\kappa =$

$\sqrt{m^2-\omega^2}$ which provides

$$L\phi = -\Lambda_0\phi, \qquad \text{for} \quad x \in \mathbb{R}.$$

Therefore,

$$\begin{bmatrix} 0 & L_- \\ -L_- & 0 \end{bmatrix} \begin{bmatrix} \phi \\ -\mathrm{i}\phi \end{bmatrix} = \Lambda\mathrm{i} \begin{bmatrix} \phi \\ -\mathrm{i}\phi \end{bmatrix},$$

and then there is an eigenvalue $\mathrm{i}\Lambda_0 \in \sigma_p\left(\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}|_{odd-even-odd-even}\right)$.

Here, we note that $\mathrm{i}\Lambda_0 = 2\omega\mathrm{i}$ is an embedded eigenvalue as long as $|\Lambda| = 2|\omega| > m - |\omega|$. Now, using all properties of $\boldsymbol{JL}$ in odd-even-odd-even subspaces as above, we consider that how perturbations affect the behavior of the embedded eigenvalues $\pm\mathrm{i}\Lambda = \pm 2\omega\mathrm{i}$. In the following theorem [3], we look for conditions for the linear instability to occur.

Theorem 2.1. *Let* $A(\epsilon) = \begin{bmatrix} 0 & L+\epsilon W_- \\ -L-\epsilon W_+ & 0 \end{bmatrix}$, $W_+ = \begin{bmatrix} c_1 & 0 \\ 0 & 2p \end{bmatrix}$, $W_- = \begin{bmatrix} c_1 & 0 \\ 0 & 2q \end{bmatrix}$ *for all* $c_1, c_2 \in \mathbb{R}$.

1. *If we suppose that* $p = q$*, then there is an open neighborhood* $\Omega \subset \mathbb{R}$, $0 \in \Omega$ *such that for* $\epsilon \in \Omega - \{0\}$ *the operator* $A(\epsilon)$ *has two eigenvalues* $\pm\mathrm{i}(2\omega + \lambda(\epsilon))$ *with* $\lambda(\epsilon) \in \mathbb{R}$ *for* $\epsilon \in \Omega - \{0\}$, $\lambda(\epsilon) \to 0$ *as* $\epsilon \to 0$.

2. *If we suppose that* $p \neq q$ *and* $p + q < 0$*, then there is an open neighborhood* $\Omega \subset \mathbb{R}$, $0 \in \Omega$ *such that for* $\epsilon \in \Omega - \{0\}$ *the operator* $A(\epsilon)$ *has four eigenvalues* $\pm\mathrm{i}(2\omega + \lambda(\epsilon))$, $\pm\mathrm{i}(2\omega + \overline{\lambda(\epsilon)})$ *with* $\operatorname{Im}\lambda(\epsilon) \neq 0$ *for* $\epsilon \in \Omega - \{0\}$, $\lambda(\epsilon) \to 0$ *as* $\epsilon \to 0$.

3. *If we suppose that* $p \neq q$ *and* $p + q > 0$*, then there is an open neighborhood* $\Omega \subset \mathbb{R}$, $0 \in \Omega$ *such that for* $\epsilon \in \Omega - \{0\}$ *the operator* $A(\epsilon)$ *has no eigenvalues in an open neighborhood of* $\pm 2\omega\mathrm{i}$.

Proof. Firstly, we need to know that the instability occurs, that is, whether there are eigenvalues with positive real part. In this proof, we look for the eigenvalues from the first quadrant.

Now, we look at the perturbed operator

$$\begin{bmatrix} 0 & L+\epsilon W_- \\ -L-\epsilon W_+ & 0 \end{bmatrix}$$

where $L = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix}$ and show that $\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}$, with $L_\pm = L + \epsilon W_\pm$, has two Jost solutions, i.e. solutions with certain decaying asymptotics, which correspond to the eigenvalue $\mathrm{i}\Lambda = \mathrm{i}(2\omega+\lambda)$ where $\operatorname{Re}\Lambda > 0$, $\operatorname{Im}\Lambda < 0$ and decay as $x \to +\infty$.

We firstly see that $\phi(x) = \begin{bmatrix} \vartheta \\ m+\Lambda-\omega \end{bmatrix} e^{-\vartheta|x|}$ where $\vartheta = \sqrt{m^2-(\Lambda-\omega)^2}$ is the Jost solution of the operator L corresponding to $-\Lambda$ due to the relation $L\phi = -\Lambda\phi$ for $x > 0$. Therefore, choosing the branch of the square root such that $\operatorname{Re}\vartheta > 0$ for $\mathrm{i}\Lambda$ from the first quadrant, we have:

$$\begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix} \begin{bmatrix} \phi \\ -\mathrm{i}\phi \end{bmatrix} = \mathrm{i}\Lambda \begin{bmatrix} \phi \\ -\mathrm{i}\phi \end{bmatrix}.$$

Now, we look for the Jost solution of L corresponding to the eigenvalue Λ as below:

$$L\zeta = \Lambda\zeta \qquad \text{where} \qquad \zeta(x) = \begin{bmatrix} \mathrm{i}\xi \\ \omega+\Lambda-m \end{bmatrix} e^{\mathrm{i}\xi|x|}$$

At this point, since there is no δ-function, we are not interested in the jump condition on ζ at

$x = 0$. Then, for $x \neq 0$, from the relation $(L - \Lambda)\zeta = 0$ we compute ξ:

$$\det \begin{bmatrix} m - \omega - \Lambda & i\xi \\ -i\xi & -m - \omega - \Lambda \end{bmatrix} = (\Lambda + \omega)^2 - m^2 - \xi^2 = 0,$$

and then $\xi = \sqrt{(\Lambda + \omega)^2 - m^2}$. In this case, we choose the branch of the square root such that $\operatorname{Im} \xi > 0$ for $\mathrm{i}\Lambda$ in the first quadrant. Then, from the equality $\xi^2 = (\Lambda + \omega)^2 - m^2$, we see that $\operatorname{Re} \xi \operatorname{Im} \xi$ has the same sign as those of $\operatorname{Im} \Lambda$, for $\omega > 0$. Choosing that $\operatorname{Re} \xi(\Lambda) < 0$ $\operatorname{Im} \xi(\Lambda) > 0$, we have:

$$\begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \mathrm{i}\xi \end{bmatrix} = \mathrm{i}\Lambda \begin{bmatrix} \xi \\ \mathrm{i}\xi \end{bmatrix},$$

for $x > 0$.

Hence, using the Jost solutions as above, we construct eigenfunctions in the invariant subspace odd-even-odd-even such that

$$\Phi(x, \Lambda) = \begin{bmatrix} \vartheta \operatorname{sgn} x \\ U(\Lambda) \\ -\mathrm{i}\vartheta \operatorname{sgn} x \\ -\mathrm{i}U(\Lambda) \end{bmatrix} e^{-\vartheta|x|}, \qquad H(x, \Lambda) = \begin{bmatrix} \mathrm{i}\xi \operatorname{sgn} x \\ V(\Lambda) \\ -\xi \operatorname{sgn} x \\ \mathrm{i}V(\Lambda) \end{bmatrix} e^{\mathrm{i}\xi|x|}$$

where

$$U(\Lambda) = m + \Lambda - \omega, \qquad V(\Lambda) = \omega + \Lambda - m.$$

Here, we consider the jump conditions by adding the perturbations ϵW_- and ϵW_+ to the matrix $\begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix}$. Due to the relation $(A(\epsilon) - \mathrm{i}\Lambda)(a\Phi(x, \Lambda) + bH(x, \Lambda)) = 0$, in more detail, we have:

$$\begin{bmatrix} -\mathrm{i}\Lambda & L_0+\delta(x)\begin{bmatrix} -2\mu & 0 \\ 0 & 2(\mu+\epsilon q) \end{bmatrix} \\ -L_0-\delta(x)\begin{bmatrix} -2\mu & 0 \\ 0 & 2(\mu+\epsilon p) \end{bmatrix} & -\mathrm{i}\Lambda \end{bmatrix} \left(a \begin{bmatrix} \vartheta \operatorname{sgn} x \\ R(\Lambda) \\ -\mathrm{i}\vartheta \operatorname{sgn} x \\ -\mathrm{i}R(\Lambda) \end{bmatrix} e^{-\vartheta|x|} + b \begin{bmatrix} \mathrm{i}\xi \operatorname{sgn} x \\ S(\Lambda) \\ -\xi \operatorname{sgn} x \\ \mathrm{i}S(\Lambda) \end{bmatrix} e^{\mathrm{i}\xi|x|} \right) = 0$$

where $L_0 = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix}$. Considering the jump conditions, we get the following equality:

$$\begin{bmatrix} 0 \\ -2(-a\mathrm{i}\vartheta + b\mathrm{i}.\mathrm{i}\xi) \\ 0 \\ 2(a\vartheta + b\mathrm{i}\xi) \end{bmatrix} + \begin{bmatrix} 0 \\ 2(\mu+\epsilon q)(-a\mathrm{i}U + b\mathrm{i}V) \\ 0 \\ -2(\mu+\epsilon p)(aU + bV) \end{bmatrix} = 0,$$

and then the system

$$-a\vartheta + b\mathrm{i}\xi + (\mu+\epsilon q)(aU - bV) = 0 \tag{2.14}$$

$$-a\vartheta - b\mathrm{i}\xi + (\mu+\epsilon p)(aU + bV) = 0. \tag{2.15}$$

We assume $p = q$. In equations (2.14) and (2.15), one can take $b = 0$ since $\xi \neq 0$. Therefore, we have the equality

$$-\vartheta + (\mu+\epsilon p)U(\Lambda) = 0$$

$$\epsilon = \frac{1}{p}\left(\frac{\vartheta}{U(\Lambda)} - \mu\right),$$

and from this relation, we arrive at the result that Λ remains purely real, i.e., the eigenvalues $\pm\mathrm{i}\Lambda$ move along imaginary axis.

Let us look at case $p \neq q$. Rewriting he equations (2.14) and (2.15) as below:

$$((\mu+\epsilon q)U(\Lambda) - \vartheta)a - (V(\Lambda)(\mu+\epsilon q) - \mathrm{i}\xi)b = 0,$$

$$((\mu+\epsilon p)U(\Lambda)-\vartheta)a+(V(\Lambda)(\mu+\epsilon p)-\mathrm{i}\xi)b=0$$

and using the compability condition of this system, we conclude that

$$\det\begin{bmatrix}(\mu+\epsilon q)U(\Lambda)-\vartheta & -(V(\Lambda)(\mu+\epsilon q)-\mathrm{i}\xi)\\ (\mu+\epsilon p)U(\Lambda)-\vartheta & V(\Lambda)(\mu+\epsilon p)-\mathrm{i}\xi\end{bmatrix}=0,$$

i.e., $((\mu+\epsilon q)U(\Lambda)-\vartheta)(V(\Lambda)(\mu+\epsilon p)-\mathrm{i}\xi)+(V(\Lambda)(\mu+\epsilon q)-\mathrm{i}\xi)((\mu+\epsilon p)U(\Lambda)-\vartheta)=0$. Therefore,

$$\epsilon^2(-2UVpq)+\epsilon(p+q)(\vartheta V+\mathrm{i}\xi U-2UV\mu)-2(\vartheta-U\mu)(\mathrm{i}\xi-V\mu)=0. \tag{2.16}$$

Now, we take the derivative of (2.16) with respect to Λ at the point $\Lambda=2\omega$ and use the condition $\epsilon|_{\Lambda=2\omega}=0$:

$$\epsilon'(p+q)(\vartheta V+\mathrm{i}\xi U-2UV\mu)-2(\vartheta'-U'\mu)(\mathrm{i}\xi-V\mu)=0,$$

and since $(\vartheta-U\mu)|_{\Lambda=2\omega}=0$ and $(\vartheta'-U'\mu)|_{\Lambda=2\omega}=-\frac{m}{k}$, we find ϵ' as below:

$$\epsilon'(p+q)(\mathrm{i}\xi U-UV\mu)-2\frac{m}{k}(\mathrm{i}\xi-V\mu)=0,$$

$$\frac{\partial\epsilon}{\partial\Lambda}\Big|_{\Lambda=2\omega}=\epsilon'=-\frac{2\frac{m}{k}(\mathrm{i}\xi-V\mu)}{(p+q)(\mathrm{i}\xi U-UV\mu)}=-\frac{2m}{(p+q)\vartheta U}. \tag{2.17}$$

We now check whether Λ leads to $e^{\mathrm{i}\xi(\Lambda)x}$ becoming exponentially decaying for $x>0$ to see whether perturbed $\mathrm{i}\Lambda(\epsilon)$ is an eigenvalue. At this point, we consider this in the case $\operatorname{Re}\Lambda\geq 0$ and $\operatorname{Im}\Lambda\leq 0$, in other words, for the eigenvalues λ in the first quadrant. For $\omega\in(0,m)$, we have $\xi^2=(\Lambda+\omega)^2-m^2$ and then we must choose the branch of the square root such that $\operatorname{Im}\leq 0$ corresponds to

$$\operatorname{Im}\xi(\Lambda)\geq 0,\qquad \operatorname{Re}\xi(\Lambda)\leq 0.$$

In this case, we need to check whether $\operatorname{Im}\Lambda < 0$ for $\epsilon \neq 0$ from an open neighborhood of $\epsilon = 0$. In order to find this condition, to have linear instability, we need to have the following conditions at the point $\epsilon = 0$:

$$\frac{\partial \operatorname{Im}\Lambda}{\partial \epsilon} = 0, \qquad \frac{\partial^2 \operatorname{Im}\Lambda}{\partial \epsilon^2} < 0. \tag{2.18}$$

In order to have the linear instability as a result, we use the following lemma:

Lemma 2. *The instability condition* (2.18) *is satisfied if and only if*

$$\frac{\partial \operatorname{Im}\epsilon}{\partial \Lambda}\Big|_{\Lambda=2\omega} \frac{\partial^2 \operatorname{Im}\epsilon}{\partial \Lambda^2}\Big|_{\Lambda=2\omega} > 0.$$

Proof. Let $g(f(z)) = z$. Then, taking the first derivative,

$$g'(f(z))f'(z) = 1,$$

and the second derivative,

$$g''(f(z))(f'(z))^2 + g'(f(z))f''(z) = 0,$$

we obtain the equality

$$g''(f(z)) = -\frac{f''(z)}{(f'(z))^3}.$$

From this, we can conclude that $\frac{\partial^2 \operatorname{Im}\Lambda}{\partial \epsilon^2}|_{\epsilon=0}$ and $\frac{\partial \operatorname{Im}\epsilon}{\partial \Lambda}|_{\Lambda=2\omega} \frac{\partial^2 \operatorname{Im}\epsilon}{\partial \Lambda^2}|_{\Lambda=2\omega}$ have opposite sign. □

Now, we proceed by taking the second derivative of (2.16) with respect to Λ at the point $\Lambda = 2\omega$. Using the condition $\epsilon|_{\Lambda=2\omega} = 0$, we have the equation

$$2(\epsilon')^2(-2UVpq) + \epsilon''(p+q)(\mathrm{i}\xi U - UV\mu) + 2\epsilon'(p+q)(\vartheta V + \mathrm{i}\xi U - 2UV\mu)'$$

$$-4(\vartheta' - \mu)(\mathrm{i}\xi - V\mu)' - 2\vartheta''(\mathrm{i}\xi - V\mu) = 0,$$

and by the equalities (2.17) and $(\vartheta' - \mu)|_{\Lambda=2\omega} = -\frac{m}{k}$, it becomes the following equation:

$$-4\left(\frac{2m}{(p+q)\vartheta U}\right)^2 UVpq + \epsilon''(p+q)(\mathrm{i}\xi U - UV\mu)$$

$$-\frac{4m}{\vartheta U}(\vartheta' V + \vartheta + i\xi' U + \mathrm{i}\xi - 2U\mu - 2V\mu) + \frac{4m}{\vartheta}(\mathrm{i}\xi' - \mu) - 2\vartheta''(\mathrm{i}\xi - V\mu) = 0.$$

Since $\vartheta|_{\Lambda=2\omega} = U\mu$, we get the equality

$$\epsilon''(p+q)U(V^2\mu^2 + \xi^2)$$

$$= (V\mu{+}\mathrm{i}\xi)\left(-4\left(\frac{2m}{(p+q)\vartheta U}\right)^2 UVpq - \frac{4m}{\vartheta U}(\vartheta' V + \vartheta + \mathrm{i}\xi - 2U\mu - 2V\mu) - \frac{4m}{\vartheta}\mu - 2\vartheta''(\mathrm{i}\xi - V\mu)\right)$$

$$= (V\mu + \mathrm{i}\xi)\left(-4\left(\frac{2m}{(p+q)\vartheta U}\right)^2 UVpq - \frac{4m}{\vartheta U}(\vartheta' V + \mathrm{i}\xi - 2V\mu) - 2\vartheta''(\mathrm{i}\xi - V\mu)\right).$$

We consider only the imaginary part of ϵ'', then we can ignore term with ϑ''. Therefore,

$$\epsilon''(p+q)U(V^2\mu^2 + \xi^2)$$

$$\sim_{mod\mathbb{R}} (V\mu + \mathrm{i}\xi)\left(-4\left(\frac{2m}{(p+q)\vartheta U}\right)^2 UVpq - \frac{4m}{\vartheta U}(\vartheta' V + \mathrm{i}\xi - 2V\mu)\right)$$

$$\sim_{mod\mathbb{R}} \mathrm{i}\xi\left(-4\left(\frac{2m}{(p+q)\vartheta U}\right)^2 UVpq - \frac{4m}{\vartheta U}(\vartheta' V - 2V\mu)\right) + V\mu\frac{4m}{\vartheta U}\mathrm{i}\xi$$

$$= \mathrm{i}\xi\frac{4mV}{\vartheta U}\left(-\frac{4mpq}{(p+q)^2\vartheta} + \frac{\omega}{\vartheta} + 3\mu\right) = \mathrm{i}\xi\frac{4m^2V}{\vartheta^2 U}\left(-\frac{4pq}{(p+q)^2} - 2\frac{\omega}{m} + 3\right).$$

Therefore, we get the inequality $-\frac{4pq}{(p+q)^2} - 2\frac{\omega}{m} + 3 > 0$ for any $p, q \in \mathbb{R}$ since $\frac{4pq}{(p+q)^2} \leq 1$ and then $\frac{\partial^2 \operatorname{Im}\epsilon}{\partial\Lambda^2}|_{\Lambda=2\omega} < 0$ by the choice $\operatorname{Re}\xi|_{\Lambda=2\omega} < 0$.

As a conclusion, by the equality (2.17) and since the inequality $\frac{\partial \operatorname{Im}\epsilon}{\partial\Lambda}|_{\Lambda=2\omega} > 0$ holds for $\epsilon \neq 0$, we have the instability condition satisfied if $p + q < 0$ and $p \neq q$.

In case $p+q>0$ and $p\neq q$, we have $\frac{\partial \operatorname{Im}\epsilon}{\partial\Lambda}|_{\Lambda=2\omega}<0$, therefore, we cannot get the instability.

$\square$

2.4. Perturbation of type $\beta+sI_2$ of the Soler model

Here, we add a small perturbation s to the Pauli matrice β and compute p and q in Theorem 2.1. From now, we use $\beta+sI_2$ instead of β in equation (2.3) as below:

$$\mathrm{i}\frac{\partial\psi}{\partial t}=-\mathrm{i}\alpha\frac{\partial\psi}{\partial x}+\beta m\psi-\delta(x)|\psi^*(\beta+sI_2)\psi|^k(\beta+sI_2)\psi \tag{2.19}$$

where $(x,t)\in\mathbb{R}\times\mathbb{R}$. We follow the same process as in Section (2.1). In other words, we consider the solitary waves in the form $\phi=a(s)\begin{bmatrix}1\\ \mu\operatorname{sgn}x\end{bmatrix}e^{-\kappa|x|}$ where $\kappa=\sqrt{m^2-\omega^2}$ and $\mu=\sqrt{\frac{m-\omega}{m+\omega}}$. If we look at the jump condition, we have:

$$2a(s)\mu=f((1+s)a^2)(1+s)a,\qquad 2\mu=(1+s)^{k+1}a(s)^{2k},\qquad a(s)=\Big(\frac{2\mu}{(1+s)^{k+1}}\Big)^{1/2k}.$$

Again, we consider the solution in the form of the Ansatz:

$$\psi(x,t)=(\phi(x)+R(x,t)+\mathrm{i}S(x,t))e^{-\mathrm{i}\omega t},\qquad \text{for}\qquad R(x,t),\ S(x,t)\in\mathbb{R}^2.$$

$$\dot R=-\mathrm{i}\alpha\frac{\partial S}{\partial x}+\beta mS-\omega S-\delta(x)f(\tau)(\beta+sI_2)S:=L_-(s)S$$

$$-\dot S=-\mathrm{i}\alpha\frac{\partial R}{\partial x}+\beta mR-\omega R-\delta(x)f(\tau)(\beta+sI_2)R-2\delta(x)|(\phi^*(\beta+sI_2)R)|f'(\tau)(\beta+sI_2)\phi:=L_+(s)R$$

where $\tau:=\phi^*(\beta+sI_2)\phi|_{x=0}=(1+s)a^2$. Here, we make a comparison between $L_\pm(s)$ and $L_\pm(s)\big|_{s=0}$ of the unperturbed Soler model:

$$W_-=L_-(s)-L_-(0)=-\delta(x)f((1+s)a(s)^2)(\beta+sI_2)+\delta(x)f(a(0)^2)\beta$$

$$= \delta(x) \begin{bmatrix} -(1+s)^k a(s)^{2k}(1+s) + a(0)^{2k} & 0 \\ 0 & (1+s)^k a(s)^{2k}(1+s) - a(0)^{2k} \end{bmatrix}$$

$$= \delta(x) \begin{bmatrix} 0 & 0 \\ 0 & (1+s)^k a(s)^{2k}(1+s) - a(0)^{2k} \end{bmatrix} = \delta(x) \begin{bmatrix} 0 & 0 \\ 0 & 2\mu\left(\frac{1-s}{1+s} - 1\right) \end{bmatrix},$$

$$W_+ = L_+(s) - L_+(0) = W_- + \delta(x) \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix},$$

with some $c \in \mathbb{R}$ which we do not use. Thus, we find

$$W_\pm = \delta(x) \begin{bmatrix} c & 0 \\ 0 & -2\mu\frac{2s}{1+s} \end{bmatrix}$$

where $c \in \mathbb{R}$ does not contribute to the spectrum when we consider the operator in the space of odd-even-odd-even functions and then, in the notations of Theorem 2.1, we have $p = q = -2\mu\frac{2s}{1+s}$. By Theorem 2.1, such a perturbation shifts the eigenvalues $\pm 2\omega\mathrm{i}$ along the imaginary axis.

3. ON THE NONLINEAR DIRAC EQUATION WITH CONCENTRATED AND SHIFTED NONLINEARITY

3.1. Solitary Waves

In this chapter, we consider the behavior of solitary wave solutions to the nonlinear Dirac equation with Soler type nonlinearity concentrated at one point by shifting the δ-function and examine how the instability differs from that in Chapter II.

Firstly, we need to write out the nonlinear Dirac equation with Soler type nonlinearity concentrated and shifted at one point in regard to the case $h \to 0$. In other words, the equation with shifted nonlinearity must be equal to the nonlinear Dirac equation with Soler type nonlinearity concentrated at one point in Chapter II as h goes to zero. Therefore, we can modify the equation by adding some coefficients in front of the nonlinearity as below:

We find solitary wave solutions to the equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\mathrm{i}\alpha\frac{\partial \psi}{\partial x} + \beta m\psi - \frac{1}{2}\delta_h(x)|\psi^*\beta\psi|^k\beta\psi - \frac{1}{2}\delta_{-h}(x)|\psi^*\beta\psi|^k\beta\psi \tag{3.1}$$

for $h > 0$. For the solitary waves in the form $\psi(x,t) = \phi(x)e^{-\mathrm{i}\omega t}$ where $(x,t) \in \mathbb{R} \times \mathbb{R}$, we have the relation

$$\omega\phi = -\mathrm{i}\alpha\frac{\partial \phi}{\partial x} + \beta m\phi - \frac{1}{2}\delta_h(x)|\phi^*\beta\phi|^k\beta\phi - \frac{1}{2}\delta_{-h}(x)|\phi^*\beta\phi|^k\beta\phi. \tag{3.2}$$

Also, we need to regard the necessity which the solitary waves must satisfy the solitary waves of the unshifted equation in Chapter II as $h \to 0$. Hence, we consider the solitary waves in the form

$$\phi_\omega = \begin{bmatrix} \frac{a}{2} + \frac{A}{2}\operatorname{sgn}(x+h) \\ \frac{b}{2}\operatorname{sgn}(x+h) \end{bmatrix} e^{-\kappa|x+h|} + \begin{bmatrix} \frac{a}{2} - \frac{A}{2}\operatorname{sgn}(x-h) \\ \frac{b}{2}\operatorname{sgn}(x-h) \end{bmatrix} e^{-\kappa|x-h|}$$

where $\kappa = \sqrt{m^2 - \omega^2}$. Again, substituting ϕ_ω into the relation (3.1), we obtain the following equation:

$$\begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} \phi_\omega = \left|\frac{|a+(a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\right|^k \frac{\delta(x+h)}{2} \begin{bmatrix} \frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa} \\ -\frac{b}{2}e^{-2h\kappa} \end{bmatrix}$$
$$+ \left|\frac{|a+(a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\right|^k \frac{\delta(x-h)}{2} \begin{bmatrix} \frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa} \\ \frac{b}{2}e^{-2h\kappa} \end{bmatrix}$$

by using the property $F(x)\delta(x-\alpha) = F(\alpha)\delta(x-\alpha)$ of the δ-function for any function $F(x)$ continuous at the point $x = \alpha$. As the terms with δ-function are cancelled in cases $x < h$ and $x > h$, choosing one of these intervals, we can find the relation between a, A and b.

If we suppose $x > h$, we have the relation

$$\begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} \left(\begin{bmatrix} \frac{a+A}{2} \\ \frac{b}{2} \end{bmatrix} e^{-\kappa(x+h)} + \begin{bmatrix} \frac{a-A}{2} \\ \frac{b}{2} \end{bmatrix} e^{-\kappa(x-h)} \right) = 0. \tag{3.3}$$

Since the first and second component of the matrix-vector product (3.3) give the same relation between a, A and b, it is sufficient to look at one of the components. We choose to consider the first component, as follows:

$$(m-\omega)\frac{a}{2}(e^{-\kappa(x+h)} + e^{-\kappa(x-h)}) + (m-\omega)\frac{A}{2}(e^{-\kappa(x+h)} - e^{-\kappa(x-h)}) - \kappa\frac{b}{2}(e^{-\kappa(x+h)} + e^{-\kappa(x-h)}) = 0,$$

$$(m-\omega)\frac{a}{2}(e^{-\kappa(x+h)} + e^{-\kappa(x-h)}) + (m-\omega)\frac{A}{2}(e^{-\kappa(x+h)} - e^{-\kappa(x-h)}) = \kappa\frac{b}{2}(e^{-\kappa(x+h)} + e^{-\kappa(x-h)}),$$

$$(m-\omega)a + (m-\omega)A\Big(\frac{e^{-\kappa(x+h)} - e^{-\kappa(x-h)}}{e^{-\kappa(x+h)} + e^{-\kappa(x-h)}}\Big) = \kappa b,$$

$$(m-\omega)a + (m-\omega)A\Big(\frac{e^{-2\kappa h} - 1}{e^{-2\kappa h} + 1}\Big) = \kappa b,$$

$$b = \frac{m-\omega}{\kappa}\left(a + A\Big(\frac{e^{-2\kappa h} - 1}{e^{-2\kappa h} + 1}\Big)\right) = \mu\left(a + A\Big(\frac{e^{-2\kappa h} - 1}{e^{-2\kappa h} + 1}\Big)\right), \tag{3.4}$$

where $\mu := \sqrt{\frac{m-\omega}{m+\omega}}$.

We now check the jump condition, comparing the coefficients at the δ-function. At this point, we have:

$$\begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} \phi_\omega = \Big|\frac{|a + (a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\Big|^k \frac{\delta(x+h)}{2} \begin{bmatrix} \frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa} \\ -\frac{b}{2}e^{-2h\kappa} \end{bmatrix}$$
$$+\Big|\frac{|a + (a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\Big|^k \frac{\delta(x-h)}{2} \begin{bmatrix} \frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa} \\ \frac{b}{2}e^{-2h\kappa} \end{bmatrix}. \tag{3.5}$$

From the relation (3.5), we obtain two following equations:

$$b = \frac{1}{2}\Big|\frac{|a + (a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\Big|^k \Big(\frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa}\Big)$$

$$A = \frac{1}{2}\Big|\frac{|a + (a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\Big|^k \frac{b}{2}e^{-2h\kappa}.$$

We denote $\chi := \Big|\frac{|a+(a+A)e^{-2h\kappa}|^2 - |b|^2 e^{-4h\kappa}}{4}\Big|^k$. Therefore, we have the relations between a, A and b as below:

$$b = \frac{\chi}{4}(a + (a+A)e^{-2h\kappa}), \tag{3.6}$$

$$A = \frac{\chi}{4} b e^{-2h\kappa}. \tag{3.7}$$

From the equalities (3.4), (3.6) and (3.7), we find the relation between χ and μ. Adding $-\frac{\chi}{4}A$ to both sides of equation (3.6) and using the relation (3.7), we get

$$b-\frac{\chi}{4}A=\frac{\chi}{4}(a+(a+A)e^{-2h\kappa})-\frac{\chi}{4}A=\frac{\chi}{4}(a(1+e^{-2h\kappa})+A(e^{-2h\kappa}-1))$$

$$b-\frac{\chi}{4}(\frac{\chi}{4}be^{-2h\kappa})=\frac{\chi}{4}(a(1+e^{-2h\kappa})+A(e^{-2h\kappa}-1))$$

$$b(1-\frac{\chi^2}{16}e^{-2h\kappa})=\frac{\chi}{4}(a(1+e^{-2h\kappa})+A(e^{-2h\kappa}-1))$$

$$b=\frac{\chi/4}{1-\frac{\chi^2}{16}e^{-2h\kappa}}(a(1+e^{-2h\kappa})+A(e^{-2h\kappa}-1)) \tag{3.8}$$

Dividing both sides of the equality in (3.8) by $1+e^{-2h\kappa}$, we arrive at the following equation:

$$\frac{b}{1+e^{-2h\kappa}}=\frac{\chi/4}{1-\frac{\chi^2}{16}e^{-2h\kappa}}\frac{(a(1+e^{-2h\kappa})+A(e^{-2h\kappa}-1))}{1+e^{-2h\kappa}}=\frac{\chi/4}{1-\frac{\chi^2}{16}e^{-2h\kappa}}\left(a+A\Big(\frac{e^{-2\kappa h}-1}{e^{-2\kappa h}+1}\Big)\right)$$

Therefore, from the relation (3.4), we obtain

$$b=\frac{\frac{\chi}{4}(1+e^{-2h\kappa})}{1-\frac{\chi^2}{16}e^{-2h\kappa}}\left(a+A\Big(\frac{e^{-2\kappa h}-1}{e^{-2\kappa h}+1}\Big)\right)=\mu\left(a+A\Big(\frac{e^{-2\kappa h}-1}{e^{-2\kappa h}+1}\Big)\right), \tag{3.9}$$

and this gives the relation

$$\mu=\frac{\frac{\chi}{4}(1+e^{-2h\kappa})}{1-\frac{\chi^2}{16}e^{-2h\kappa}}. \tag{3.10}$$

Moreover, when we plug the solution in the form of the Ansatz $\psi(x,t)=(\phi_\omega(x)+\rho(x,t))e^{-\mathrm{i}\omega t}$, with $\rho(x,t)=R(x,t)+\mathrm{i}S(x,t)$ into equation (3.1), we get:

$$\frac{\partial\boldsymbol{\rho}}{\partial t}=\boldsymbol{J}\boldsymbol{L}\boldsymbol{\rho}, \qquad \boldsymbol{\rho}=\begin{bmatrix}\operatorname{Re}\rho\\ \operatorname{Im}\rho\end{bmatrix},$$

where

$$\boldsymbol{L}=\boldsymbol{J}\boldsymbol{\alpha}\frac{\partial\boldsymbol{\rho}}{\partial x}+\boldsymbol{\beta}\boldsymbol{\rho}+(m-\delta_h(x)\frac{f}{2}-\delta_{-h}(x)\frac{f}{2})\boldsymbol{\beta}\boldsymbol{\rho}-\delta_h(x)(\boldsymbol{\phi}^*\beta\rho)f'\boldsymbol{\beta}\boldsymbol{\phi}-\delta_{-h}(x)(\boldsymbol{\phi}^*\beta\rho)f'\boldsymbol{\beta}\boldsymbol{\phi}-\omega\boldsymbol{\rho}$$

for $f := |\phi^*\beta\phi|^k$, with

$$\boldsymbol{\alpha} = \begin{bmatrix} \operatorname{Re}\alpha & -\operatorname{Im}\alpha \\ \operatorname{Im}\alpha & \operatorname{Re}\alpha \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \operatorname{Re}\beta & -\operatorname{Im}\beta \\ \operatorname{Im}\beta & \operatorname{Re}\beta \end{bmatrix}, \quad \boldsymbol{\phi} = \begin{bmatrix} \operatorname{Re}\phi \\ \operatorname{Im}\phi \end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}.$$

Therefore, we find that the linearization at a solitary wave is represented by the operator

$$\boldsymbol{J}\boldsymbol{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix},$$

where

$$L_- = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - \frac{\chi}{2}\delta_h(x)\sigma_3 - \frac{\chi}{2}\delta_{-h}(x)\sigma_3,$$

$$L_+ = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - \frac{\chi}{2}\delta_h(x)\sigma_3 - \frac{\chi}{2}\delta_{-h}(x)\sigma_3 - k\chi\delta_h(x)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - k\chi\delta_{-h}(x)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let us suppose that $x \neq \pm h$. From the relation $\boldsymbol{J}\boldsymbol{L}\Psi = \lambda\Psi$, we conclude that the eigenvector $\Psi(x)$ corresponding to the eigenvalue $\lambda = \mathrm{i}\Lambda$ has the form

$$\Psi(x) = A\left(\begin{bmatrix} M\operatorname{sgn}(x+h) \\ m-\omega-\Lambda \\ \mathrm{i}M\operatorname{sgn}(x+h) \\ \mathrm{i}(m-\omega-\Lambda) \end{bmatrix} e^{-M|x+h|} + \begin{bmatrix} M\operatorname{sgn}(x-h) \\ m-\omega-\Lambda \\ \mathrm{i}M\operatorname{sgn}(x-h) \\ \mathrm{i}(m-\omega-\Lambda) \end{bmatrix} e^{-M|x-h|}|\right)$$

$$+B\left(\begin{bmatrix} N\,\mathrm{sgn}(x+h) \\ m-\omega+\Lambda \\ -\mathrm{i}N\,\mathrm{sgn}(x+h) \\ -\mathrm{i}(m-\omega+\Lambda) \end{bmatrix} e^{-N|x+h|} + \begin{bmatrix} N\,\mathrm{sgn}(x-h) \\ m-\omega+\Lambda \\ -\mathrm{i}N\,\mathrm{sgn}(x-h) \\ -\mathrm{i}(m-\omega+\Lambda) \end{bmatrix} e^{-N|x-h|}\right)$$

and $M = \sqrt{m^2-(\omega+\Lambda)^2}$, $N = \sqrt{m^2-(\omega-\Lambda)^2}$.

As we did in Chapter II, one can find the spectrum of the linearization operator by considering the jump condition, equating the coefficients at the δ-function in $(\boldsymbol{JL}-\mathrm{i}\Lambda)\Psi(x)$.

3.2. Perturbation of type $\beta + sI_2$ of the Soler model

Now, we consider the nonlinear Dirac equation with Soler type nonlinearity concentrated and shifted at one point with $\beta + sI_2$ instead of β in equation (3.1) as below:

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\mathrm{i}\alpha\frac{\partial\psi}{\partial x} + \beta m\psi - \frac{1}{2}\delta_h(x)|\psi^*(\beta+sI_2)\psi|^k(\beta+sI_2)\psi - \frac{1}{2}\delta_{-h}(x)|\psi^*(\beta+sI_2)\psi|^k(\beta+sI_2)\psi \tag{3.11}$$

where $(x,t) \in \mathbb{R}\times\mathbb{R}$.

Here, we look at the solitary waves in the following form

$$\phi = \begin{bmatrix} \frac{a(s)}{2} + \frac{A(s)}{2}\,\mathrm{sgn}\,(x+h) \\ \frac{b(s)}{2}\,\mathrm{sgn}(x+h) \end{bmatrix} e^{-\kappa|x+h|} + \begin{bmatrix} \frac{a(s)}{2} - \frac{A(s)}{2}\,\mathrm{sgn}(x-h) \\ \frac{b(s)}{2}\,\mathrm{sgn}(x-h) \end{bmatrix} e^{-\kappa|x-h|}$$

where $\kappa = \sqrt{m^2-\omega^2}$.

From now, we denote $a = a(s)$, $A = A(s)$, and $b = b(s)$. If we consider the jump condition, we have

$$\begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} \phi = \frac{\delta_{-h}(x)}{2} \left| \frac{((1+s)|a+(a+A)e^{-2h\kappa}|^2-(1-s)|b|^2e^{-4h\kappa})}{4} \right|^k (\beta + sI_2) \begin{bmatrix} \frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa} \\ -\frac{b}{2}e^{-2h\kappa} \end{bmatrix}$$

$$+ \frac{\delta_h(x)}{2} \left| \frac{((1+s)|a+(a+A)e^{-2h\kappa}|^2-(1-s)|b|^2e^{-4h\kappa})}{4} \right|^k (\beta + sI_2) \begin{bmatrix} \frac{a}{2} + (\frac{a}{2} + \frac{A}{2})e^{-2h\kappa} \\ \frac{b}{2}e^{-2h\kappa} \end{bmatrix}. \tag{3.12}$$

If we simplify the equality in (3.12), we get the following system:

$$b(s) = \left| \frac{((1+s)|a+(a+A)e^{-2h\kappa}|^2-(1-s)|b|^2e^{-4h\kappa})}{4} \right|^k (1+s)(a+(a+A)e^{-2h\kappa})$$

$$= \frac{1}{4}\theta(s)(1+s)(a+(a+A)e^{-2h\kappa}) \tag{3.13}$$

$$A(s) = \left| \frac{((1+s)|a+(a+A)e^{-2h\kappa}|^2-(1-s)|b|^2e^{-4h\kappa})}{4} \right|^k (s-1)be^{-2h\kappa} = \frac{1}{4}\theta(s)(s-1)be^{-2h\kappa} \tag{3.14}$$

where $\theta(s) := \left| \frac{((1+s)|a+(a+A)e^{-2h\kappa}|^2-(1-s)|b|^2e^{-4h\kappa})}{4} \right|^k$.

In order to find the relation between μ and $\theta(s)$ in this case, we need to use the relations (3.9), (3.13) and (3.14). Subtracting $\frac{1}{4}\theta(s)(1+s)A$ from both sides of equation (3.13), we have:

$$b - \frac{1}{4}\theta(s)(1+s)A = \frac{1}{4}\theta(s)(1+s)(a+(a+A)e^{-2h\kappa}) - \frac{1}{4}\theta(s)(1+s)A$$

$$= \frac{1}{4}\chi(1+s)^{k+1}(a(e^{-2h\kappa}+1) + A(e^{-2h\kappa}-1)).$$

From (3.9), we know

$$\frac{b(e^{-2h\kappa}+1)}{\mu} = a(e^{-2h\kappa}+1) + A(e^{-2h\kappa}-1).$$

Therefore,

$$b - \frac{1}{4}\theta(s)(1+s)A = \frac{1}{4}\theta(s)(1+s)\frac{b(e^{-2h\kappa}+1)}{\mu},$$

and using the equation in (3.14), we get:

$$\begin{aligned} b - \frac{1}{4}\theta(s)(1+s)\left(\frac{1}{4}\theta(s)(s-1)be^{-2h\kappa}\right) &= \frac{1}{4}\theta(s)(1+s)\frac{b(e^{-2h\kappa}+1)}{\mu} \\ b(1-\frac{\theta^2}{16}(s^2-1)e^{-2h\kappa}) &= \frac{\theta(s)}{4}\frac{b(1+s)(e^{-2h\kappa}+1)}{\mu} \\ 1-\frac{\theta(s)^2}{16}(s^2-1)e^{-2h\kappa} &= \frac{\theta(s)}{4}\frac{(1+s)(e^{-2h\kappa}+1)}{\mu}. \end{aligned}$$

Hence,

$$\mu = \frac{\frac{\theta(s)}{4}(1+s)(e^{-2h\kappa}+1)}{1-\frac{\theta(s)^2}{16}(s^2-1)e^{-2h\kappa}}. \tag{3.15}$$

Moreover, we consider the solution in the form of the Ansatz

$$\psi(x,t) = (\phi(x) + R(x,t) + \mathrm{i}S(x,t))e^{-\mathrm{i}\omega t}, \qquad \text{for} \qquad R(x,t), S(x,t) \in \mathbb{R}^2,$$

we have that for $\rho = R + \mathrm{i}S$,

$$\dot{R} = -\mathrm{i}\alpha\frac{\partial S}{\partial x} + \beta m S - \omega S - \frac{1}{2}\delta_h(x)f(\tau)(\beta+sI_2)S - \frac{1}{2}\delta_{-h}(x)f(\tau)(\beta+sI_2)S := L_-(s)S$$

$$-\dot{S} = -\mathrm{i}\alpha\frac{\partial R}{\partial x} + \beta m R - \omega R - \frac{1}{2}\delta_h(x)f(\tau)(\beta+sI_2)R - \frac{1}{2}\delta_{-h}(x)f(\tau)(\beta+sI_2)R$$

$$-\delta_h(x)\operatorname{Re}(\phi^*(\beta+sI_2)\rho)f'(\tau)(\beta+sI_2)\phi - \delta_{-h}(x)\operatorname{Re}(\phi^*(\beta+sI_2)\rho)f'(\tau)(\beta+sI_2)\phi := -L_+(s)R$$

where $f(\tau) = \tau^k$ is evaluated at $\tau := \phi^*(\beta+sI_2)\phi|_{x=0} = \left|\frac{((1+s)|a+(a+A)e^{-2h\kappa}|^2-(1-s)|b|^2e^{-4h\kappa})}{4}\right|$. Therefore, we represent operators L_- and L_+ as below:

$$L_-(s) = -\mathrm{i}\alpha\partial_x + \beta m - \omega - \frac{1}{2}\delta_h(x)f(\tau)(\beta+sI_2) - \frac{1}{2}\delta_{-h}(x)f(\tau)(\beta+sI_2)$$

$$-L_+(s) = -\mathrm{i}\alpha\partial_x + \beta m - \omega - \frac{1}{2}\delta_h(x)f((1+s)\tau)(\beta + sI_2) - \frac{1}{2}\delta_{-h}(x)f(\tau)(\beta + sI_2)$$

$$-\delta_h(x)(\phi^*(\beta + sI_2)\rho)f'(\tau)(\beta + sI_2)\phi - \delta_{-h}(x)(\phi^*(\beta + sI_2)\rho)f'(\tau)(\beta + sI_2)\phi.$$

We note that $\phi^*(\beta + sI_2)\rho$ represents the product of the first component of the solitary wave ϕ and the real part of ρ at the origin. Hence, the linearized equation on ρ of perturbed equation at a solitary wave can be written as below:

$$\partial_t \boldsymbol{\rho} = \boldsymbol{J}\boldsymbol{L}\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix}$$

where $\boldsymbol{J}\boldsymbol{L} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}$ with

$$L_- = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - \frac{\theta(s)}{2}\delta_h(x)\sigma_3 - \frac{\theta(s)}{2}\delta_{-h}(x)\sigma_3,$$

$$L_+ = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - \frac{\theta(s)}{2}\delta_h(x)\sigma_3 - \frac{\theta(s)}{2}\delta_{-h}(x)\sigma_3 - k\theta(s)\big(\delta_h(x) + \delta_{-h}(x)\big)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, we make a comparison between $L_-(s)$ and $L_-(s)\big|_{s=0}$ and between $L_+(s)$ and $L_+(s)\big|_{s=0}$ of the unperturbed Soler model. Let us start by comparing $L_-(s)$ and $L_-(s)\big|_{s=0}$:

$$W_-(s) := L_-(s) - L_-(0)$$

$$= -\frac{(\delta_h(x) + \delta_{-h}(x))}{2}\theta(s)(\sigma_3 + sI_2) + \frac{(\delta_h(x) + \delta_{-h}(x))}{2}\theta(0)\sigma_3$$

$$= \frac{(\delta_h(x)+\delta_{-h}(x))}{2} \begin{bmatrix} -\theta(s)(1+s)+\theta(0) & 0 \\ 0 & \theta(s)(1-s)-\theta(0) \end{bmatrix}$$

$$= \frac{(\delta_h(x)+\delta_{-h}(x))}{2} \begin{bmatrix} 0 & 0 \\ 0 & \theta(s)(1-s)-\theta(0) \end{bmatrix}.$$

Using the relation between μ and $\theta(s)$ in (3.15), we have:

$$W_-(s) = \frac{(\delta_h(x)+\delta_{-h}(x))}{2} \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{4\mu(1-\frac{\theta(s)^2}{16}(s^2-1)e^{-2h\kappa})}{e^{-2h\kappa}+1}\frac{1-s}{1+s} + \frac{4\mu(1-\frac{\theta(0)^2}{16}(s^2-1)e^{-2h\kappa})}{e^{-2h\kappa}+1}\right) \end{bmatrix}$$

$$= (\delta_h(x)+\delta_{-h}(x)) \begin{bmatrix} 0 & 0 \\ 0 & \frac{2\mu}{e^{-2h\kappa}+1}\left(\left(1-\frac{\theta(s)^2}{16}(s^2-1)e^{-2h\kappa}\right)\frac{1-s}{1+s} + \left(1-\frac{\theta(0)^2}{16}e^{-2h\kappa}\right)\right) \end{bmatrix}$$

$$= (\delta_h(x)+\delta_{-h}(x)) \begin{bmatrix} 0 & 0 \\ 0 & \frac{2\mu}{e^{-2h\kappa}+1}\left(\gamma(s)\frac{1-s}{1+s} - \gamma(0)\right) \end{bmatrix},$$

where $\gamma(s) := 1 - \frac{\theta(s)^2}{16}(s^2-1)e^{-2h\kappa}$.

We now write out $W_+(s) = L_+(s) - L_+(0)$ as below:

$$W_+ = L_+(s) - L_+(0) = W_-(s) - (\delta_h(x)+\delta_{-h}(x))f'(\theta(s)^{1/k})|\phi^*(\beta+sI_2)\phi| \begin{bmatrix} 1+s & 0 \\ 0 & 0 \end{bmatrix}$$

$$+(\delta_h(x)+\delta_{-h}(x))f'(\theta(0)^{1/k})|\phi^*\beta\phi| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= W_-(s) + (\delta_h(x)+\delta_{-h}(x)),$$

$$\begin{bmatrix} -k\theta(s)^{\frac{k-1}{k}}\theta(s)^{\frac{1}{k}}(1+s)+k\theta(0)^{\frac{k-1}{k}}\theta(0)^{\frac{1}{k}} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= W_-(s) + (\delta_h(x)+\delta_{-h}(x)) \begin{bmatrix} -k(1+s)\theta(s)+k\theta(0) & 0 \\ 0 & 0 \end{bmatrix}$$

$$= W_-(s) + (\delta_h(x)+\delta_{-h}(x)) \begin{bmatrix} -k\frac{4\mu}{e^{-2h\kappa}+1}(\gamma(s)(1+s)-\gamma(0)) & 0 \\ 0 & 0 \end{bmatrix}$$

$$= (\delta_h(x)+\delta_{-h}(x)) \begin{bmatrix} \frac{4\mu k}{e^{-2h\kappa}+1}(\gamma(s)(1+s)-\gamma(0)) & 0 \\ 0 & \frac{2\mu}{e^{-2h\kappa}+1}\left(\gamma(s)\frac{1-s}{1+s}-\gamma(0)\right) \end{bmatrix}.$$

Therefore, we arrive at

$$W_\pm(s) = (\delta_h(x)+\delta_h(x)) \begin{bmatrix} c & 0 \\ 0 & \frac{2\mu}{e^{-2h\kappa}+1}\left(\gamma(s)\frac{1-s}{1+s}-\gamma(0)\right) \end{bmatrix},$$

and by Theorem 2.1, we obtain $p = q = \frac{2\mu}{e^{-2h\kappa}+1}\left(\gamma(s)\frac{1-s}{1+s}-\gamma(0)\right)$ and $c = 0$ in W_-; $c = \frac{4\mu k}{e^{-2h\kappa}+1}(\gamma(s)(1+s)-\gamma(0))$ in W_+.

Furthermore, in order study to how the eigenvalues' behavior changes, one can consider a perturbed operator: for a small parameter s,

$$\begin{bmatrix} 0 & L+sW_- \\ -L-sW_+ & 0 \end{bmatrix},$$

where $L = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix}$ by finding Jost solutions of $\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}$ and then constructing eigenfunctions in the invariant subspace odd-even-odd-even.

4. ON THE MASSIVE THIRRING MODEL WITH CONCENTRATED NONLINEARITY

In this chapter, we consider the Dirac equation with the nonlinearity from the massive Thirring model [21], again concentrated at a point just like in Chapter II. We would like to check whether the spectrum of linearization operator in such a model is different from that in Chapter II.

4.1. Solitary Waves

Here, we introduce the massive Thirring model with nonlinearity concentrated at one point:

$$\mathrm{i}\frac{\partial \Phi}{\partial t} = -\mathrm{i}\alpha\frac{\partial \Phi}{\partial x} + \beta m\Phi - \delta(x)|s|^{k-1}(J^0 - J^1\alpha)\Phi, \tag{4.1}$$

where $s = \sqrt{(J^0)^2 - (J^1)^2}$ represents the length of the charge-current density with $J^0 = \Phi^*\Phi$ and $J^1 = \Phi^*\alpha\Phi$ which are respectively charge and current densities. For the solitary wave solutions $\Phi(x,t) = \phi(x)e^{-\mathrm{i}\omega t}$ where $(x,t) \in \mathbb{R}\times\mathbb{R}$ and $\omega \in (-m, m)$, the above turns into the following equation:

$$\omega\phi = -\mathrm{i}\alpha\frac{\partial \phi}{\partial x} + \beta m\phi - \delta(x)|\phi^*\phi|^{k-1}(\phi^*\phi)\phi. \tag{4.2}$$

We note that $J^1 = 0$ on the solitary waves. Choosing $\alpha = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we study on the solitary waves for

$\phi_\omega = \begin{bmatrix} a \\ b\,\mathrm{sgn}\,x \end{bmatrix} e^{-\kappa|x|}$ where $\kappa = \sqrt{m^2-\omega^2}$. We note that the product $\delta(x)\,\mathrm{sgn}\,x$ is identically zero. At this point, we have the relation

$$\begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} \begin{bmatrix} a \\ b\,\mathrm{sgn}\,x \end{bmatrix} e^{-\kappa|x|} = |a|^{2k}\delta(x) \begin{bmatrix} a \\ 0 \end{bmatrix} e^{-\kappa|x|}. \tag{4.3}$$

In case $x > 0$, we have the system $\begin{bmatrix} m-\omega & -\kappa \\ \kappa & -m-\omega \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$, and then we can find the relation between a and b from the first component of the matrix-vector product as below:

$$b = \frac{\kappa}{m+\omega} = \mu a, \qquad \text{where} \qquad \mu := \sqrt{\frac{m-\omega}{m+\omega}}.$$

Taking into account that the derivative of the sign function at the point x is $2\delta(x)$, we equate the coefficients of $\delta(x)$ in (4.3), which gives the jump condition

$$2b = |a|^{2k}a, \qquad \text{and then} \qquad \mu = \frac{|a|^{2k}}{2}. \tag{4.4}$$

4.2. Spectrum of the linearization operator

As we did in Chapter II, we firstly study on the linearization at a solitary wave using an small perturbation ρ which is depends on the time, and then find the spectrum of the linearization operator in odd-even-odd-even subspaces.

We begin with the analysis of equation (4.1) using the solution in the form of the Ansatz

$$\Phi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-\mathrm{i}\omega t}$$

where ρ is the linearized equation such that $\rho(x,t) = R(x,t) + \mathrm{i}S(x,t)$. When we plug the solution of the form of the Ansatz in equation (4.1), we obtain the equality

$$\omega\phi_\omega + wR + \mathrm{i}wS + \mathrm{i}\dot{R} - \dot{S} = -\mathrm{i}\alpha\frac{\partial\phi_\omega}{\partial x} - \mathrm{i}\alpha\frac{\partial R}{\partial x} - \mathrm{i}\alpha\frac{\partial iS}{\partial x} + \beta m\phi_\omega + \beta mR + i\beta mS$$

$$-\delta(x)f(\phi_\omega + R + \mathrm{i}S)(\phi_\omega + R + \mathrm{i}S)$$

where $f(\phi) = |\phi^*\phi|^{2k-1}$. We can make an approximation as below:

$$|(\phi_\omega + R + \mathrm{i}S)^*(\phi_\omega + R + \mathrm{i}S)| \approx |\phi_\omega{}^*\phi_\omega + 2\phi_\omega^* R|.$$

Hence, we get the following system by classifying imaginary and real parts:

$$\dot{R} = -\mathrm{i}\alpha\frac{\partial S}{\partial x} + \beta mS - \delta(x)f(\phi_\omega)S - \omega S, \tag{4.5}$$

$$-\dot{S} = -\mathrm{i}\alpha\frac{\partial R}{\partial x} + \beta mR - \delta(x)f(\phi_\omega)R - 2\delta(x)|\phi_\omega^* R|f'\phi_\omega - \omega R, \tag{4.6}$$

where $f(\tau) = |\tau|^k$ is evaluated at $\tau = |\phi_\omega^*\phi_\omega|$. Using equations (4.5) and (4.6), we obtain the linearized equation on ρ given by

$$\frac{\partial\boldsymbol{\rho}}{\partial t} = \boldsymbol{J}\boldsymbol{L}\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \begin{bmatrix}\operatorname{Re}\rho \\ \operatorname{Im}\rho\end{bmatrix},$$

where

$$\boldsymbol{L}\boldsymbol{\rho} = \boldsymbol{J}\boldsymbol{\alpha}\frac{\partial\boldsymbol{\rho}}{\partial x} + \boldsymbol{\beta}m\boldsymbol{\rho} - \delta(x)f\boldsymbol{\rho} - 2\delta(x)(\boldsymbol{\phi}^*\rho)f'\boldsymbol{\phi} - \omega\boldsymbol{\rho}$$

with

$$\boldsymbol{\alpha} = \begin{bmatrix}\operatorname{Re}\alpha & -\operatorname{Im}\alpha \\ \operatorname{Im}\alpha & \operatorname{Re}\alpha\end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix}\operatorname{Re}\beta & -\operatorname{Im}\beta \\ \operatorname{Im}\beta & \operatorname{Re}\beta\end{bmatrix}, \quad \boldsymbol{\phi} = \begin{bmatrix}\operatorname{Re}\phi \\ \operatorname{Im}\phi\end{bmatrix}, \quad \boldsymbol{J} = \begin{bmatrix}0 & I_2 \\ -I_2 & 0\end{bmatrix}$$

and

$$\boldsymbol{L} = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

where

$$L_- = -\mathrm{i}\alpha\partial_x + \beta m - \delta(x) f(\phi_\omega) - \omega,$$

$$-L_+ = -\mathrm{i}\alpha\partial_x + \beta m - \delta(x) f(\phi_\omega) - 2\delta(x)\phi_\omega^*\rho f'\phi_\omega - \omega.$$

Here, we also note that $\phi_\omega^*\rho$ represents the product of the first component of the solitary wave ϕ_ω and the real part of ρ at the point 0.

4.2.1. The Spectrum of JL in odd-even-odd-even subspace

We look for the spectrum of the linearization operator on odd-even-odd-even subspace. In the previous section, we found that the linearization at a solitary wave is

$$\boldsymbol{JL} = \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}$$

where

$$L_- = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - 2\mu\delta(x) I_2,$$

$$L_+ = \begin{bmatrix} m-\omega & \partial_x \\ -\partial_x & -m-\omega \end{bmatrix} - 2\mu\delta(x) I_2 - 4k\mu\delta(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We now suppose that $x \neq 0$. Hence, we can conclude that if the relation $\boldsymbol{JL}\Psi = \lambda\Psi$ holds,the eigenvector $\Psi(x)$ is corresponding to the eigenvalue $\lambda = \mathrm{i}\Lambda$ where

$$\Psi(x) = A \begin{bmatrix} M \operatorname{sgn} x \\ m - \omega - \Lambda \\ \mathrm{i}M \operatorname{sgn} x \\ i(m - \omega - \Lambda) \end{bmatrix} e^{-M|x|} + B \begin{bmatrix} N \operatorname{sgn} x \\ m - \omega + \Lambda \\ -iN \operatorname{sgn} x \\ -i(m - \omega + \Lambda) \end{bmatrix} e^{-N|x|}$$

and $M = \sqrt{m^2 - (\omega + \Lambda)^2}$, $N = \sqrt{m^2 - (\omega - \Lambda)^2}$.

When we look at the jump conditions in $(\boldsymbol{JL} - \lambda)\Psi = 0$, we get the following system:

$$-MA + NB - \mu(A(m - \omega - \Lambda) - B(m - \omega + \Lambda)) = 0,$$

$$MA + NB + \mu(A(m - \omega - \Lambda) + B(m - \omega + \Lambda)) = 0,$$

and in this system, we can see that there is a change in the signs in front of μ compared to the jump conditions in Chapter II. Therefore, we have:

$$\begin{bmatrix} -M - \mu(m - \omega - \Lambda) & N + \mu(m - \omega + \Lambda) \\ M + \mu(m - \omega - \Lambda) & N + \mu(m - \omega + \Lambda) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we have the compatibility condition

$$\det \begin{bmatrix} -M - \mu(m - \omega - \Lambda) & N + \mu(m - \omega + \Lambda) \\ M + \mu(m - \omega - \Lambda) & N + \mu(m - \omega + \Lambda) \end{bmatrix}$$

$$= -2(M + \mu(m - \omega - \Lambda))(N + \mu(m - \omega + \Lambda)) = 0.$$

Therefore, at least one of the relations $M + \mu(m - \omega - \Lambda) = 0$, $N + \mu(m - \omega + \Lambda) = 0$ hold.

If $M + \mu(m - \omega - \Lambda) = 0$, then we have:

$$m^2 - (\omega + \Lambda)^2 = \frac{m - \omega}{m + \omega}(m - \omega - \Lambda)^2,$$

$$m + \omega + \Lambda = \frac{m - \omega}{m + \omega}(m - \omega - \Lambda),$$

$$(m + \omega + \Lambda)(m + \omega) = (m - \omega)(m - \omega - \Lambda),$$

$$(m + \omega)^2 - (m - \omega)^2 = -2m\Lambda,$$

$$\Lambda = -2\omega.$$

In the case $N + (m - \omega + \Lambda) = 0$, in a similar way, we get $\Lambda = 2\omega$. Therefore, we have the eigenvector $\Psi(x)$ corresponding to the eigenvalue $\lambda = \pm 2\omega \mathrm{i}$ which are present in the spectrum of $\boldsymbol{JL}$. As a conclusion, despite the change in the signs of μ in the jump conditions, we arrive at the same eigenvalues as the ones in Chapter II.

5. CONCLUSION

In Chapter II, we analyzed the structure of solitary waves of the nonlinear Dirac equation with concentrated nonlinearity. Moving on to Chapter III, we constructed solitary waves of the nonlinear Dirac equation with concentrated and shifted nonlinearity by having regard to the fact that the relations must satisfy those of solitary waves in Chapter II as $h \to 0$.

Furthermore, in Chapter II and Chapter IV, we looked for the spectrum of linearization of the nonlinear Dirac equation with concentrated and shifted nonlinearity and the massive Thirring model at solitary waves in odd-even-odd-even subsaces and we saw that they have the eigenfunctions of the same structure corresponding to the eigenvalues $\lambda = \pm 2\omega \mathrm{i}$.

Moreover, in Chapter II, we laid the ground for analyzing the behavior of bifurcations from $2\omega\mathrm{i}$ under perturbations of the nonlinearity when SU(1,1) symmetry is broken.

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